Clock Coefficients from Allan Variance Diagram

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1 GPS One-Way Range Measurements

GPS one-way range (and Doppler) measurements are defined by two clocks (oscillators), a GPS NAVSTAR atomic clock (Cesium or Rubidium) and a USER spacecraft clock (usually a temperature controlled crystal clock). The time of GPS NAVSTAR radio wave front emission is defined by a NAVSTAR clock, and time of USER reception is defined by the USER crystal clock. One-way range is defined by the time difference. The fractional frequency stability of the USER crystal clock is usually inferior to that of every NAVSTAR clock, so the stochastic phenomenology of the USER crystal clock is exposed by each NAVSTAR clock.

When simultaneous USER-GPS range measurements are differenced, then the stochastic phenomenology of the USER crystal clock is differenced out[1], and can be ignored in the orbit determination using real range measurements. But otherwise, it must be accounted for in the orbit determination. The stochastic phenomenology of the USER crystal clock should be accounted for in all *simulations* of one-way range.

The manufacturer of a clock usually provides an Allan Variance Diagram, for an associated class of clocks, to describe the fractional frequency stability of that clock. One-way range measurement errors can be characterized and simulated, in part, using the fractional frequency stability of the inferior clock. Values for Allan clock parameters a_0 and a_{-2} are required for stochastic clock models to be used for simulation and orbit determination. The purpose of this note is to describe a method for derivation of crystal clock values of a_0 and a_{-2} from an Allan Variance Diagram. Then the simulation of the stochastic phenomenology of a crystal clock can be constructed from values of a_0 and a_{-2}.

2 Values of a_0 and a_{-2} from a Simulated Allan Variance Diagram

The Allan Variance Diagram presents the logarithm (base 10) of clock sample time τ (sec) on the abscissa (x axis), and the logarithm (base 10) of the square-root $\sigma_y(\tau)$ of the Allan Variance $\sigma_y^2(\tau)$ on the ordinate (y axis). That is, $\log_{10}\sigma_y(\tau)$ is graphed as a function of $\log_{10}\tau$.

Fig. 1 (Chuba[2]) presents an ideal simulated Allan Variance Diagram. Abscissa values for Fig. 1 are $\{0, 1, 2, 3, 4, 5\}$. I have replaced this set with $\{10^0, 10^1, 10^2, 10^3, 10^4, 10^5\}$, to enable identification of τ (sec) directly from the diagram. Ordinate values for Fig. 1 are $\{-10, -11, -12, -13, -14\}$. I have replaced this set with $\{10^{-10}, 10^{-11}, 10^{-12}, 10^{-13}, 10^{-14}\}$, to enable identification of $\sigma_y(\tau)$ directly from the diagram.

From Allan[1]:

$$\sigma_y^2(\tau) = \begin{cases} a_0\tau^{-1} & \text{frequency white noise} \\ a_{-1}\tau^{0} & \text{frequency flicker noise} \\ a_{-2}\tau^{1} & \text{frequency random walk} \end{cases} \tag{1}$$

[1] And relativistic effects are differenced out.

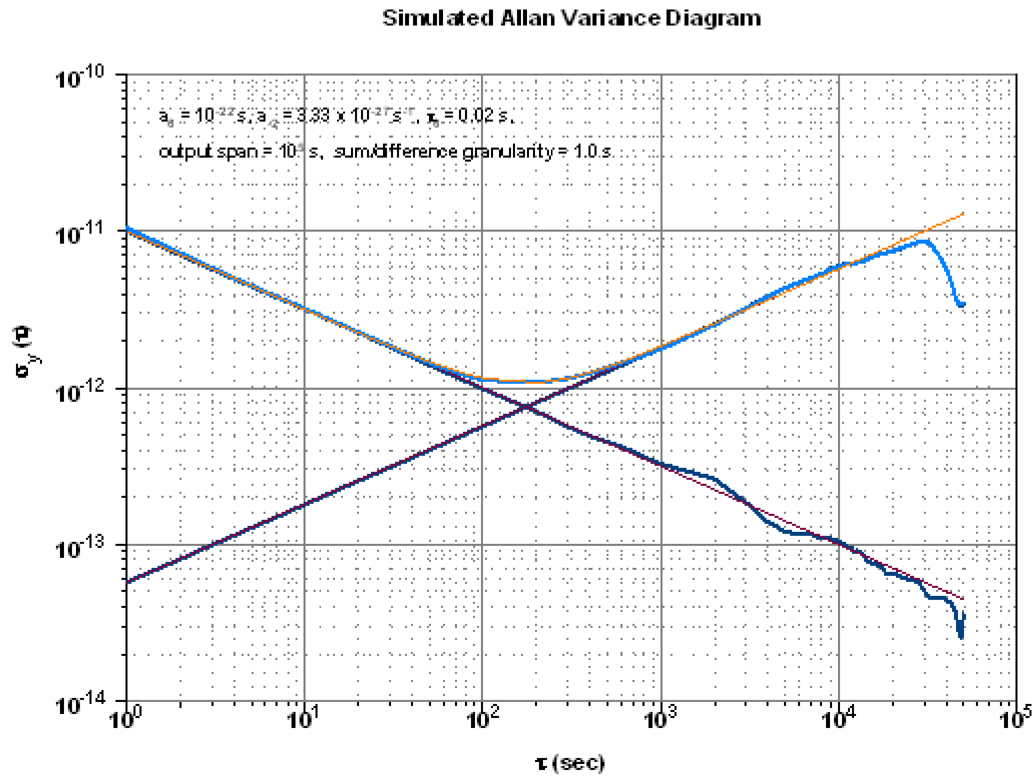

Figure 1: Simulated Allan Variance Diagram

Thus a_0 and a_{-2} are associated with clock frequency white noise and clock frequency random walk. For crystal clocks I shall ignore frequency flicker noise[2].

The Allan root-variance graph approximation from Fig. 1 is defined by the sum of the two straight lines. Frequency white noise (for a_0) is associated with the straight line with negative slope $(-1/2)$. Frequency random walk (for a_{-2}) is associated with the straight line with positive slope $(1/2)$. The wiggly overlaid functions are reconstructions from simulated data – they become poorer approximations as τ increases because the sample ensemble is exhausted (becomes smaller) as τ increases.

2.1 Calculate Values of a_0 and a_{-2}

Each of the following examples derives from one point read from Fig. 1.

2.1.1 Examples for a_0

$$a_0 = [\tau]\left[\sigma_y^2(\tau)\right] = \left[1.0 \times 10^4 \,\text{sec}\right]\left[\left(1.00 \times 10^{-13}\right)^2\right] = 1.0 \times 10^{-22}\,\text{sec}$$

$$a_0 = [\tau]\left[\sigma_y^2(\tau)\right] = \left[1.0 \times 10^0 \,\text{sec}\right]\left[\left(1.00 \times 10^{-11}\right)^2\right] = 1.0 \times 10^{-22}\,\text{sec}$$

$$a_0 = [\tau]\left[\sigma_y^2(\tau)\right] = \left[1.0 \times 10^1 \,\text{sec}\right]\left[\left(3.17 \times 10^{-12}\right)^2\right] = 1.0 \times 10^{-22}\,\text{sec}$$

[2] Flicker noise is significant in atomic clocks.

2.1.2 Examples for a_{-2}

$$a_{-2} = \left[\tau^{-1}\right]\left[\sigma_y^2\left(\tau\right)\right] = \left[\left(3.0 \times 10^4 \,\text{sec}\right)^{-1}\right]\left[\left(1.00 \times 10^{-11}\right)^2\right] = 3.33 \times 10^{-27} \,\text{sec}^{-1}$$

$$a_{-2} = \left[\tau^{-1}\right]\left[\sigma_y^2\left(\tau\right)\right] = \left[\left(1.0 \times 10^0 \,\text{sec}\right)^{-1}\right]\left[\left(5.77 \times 10^{-14}\right)^2\right] = 3.33 \times 10^{-27} \,\text{sec}^{-1}$$

3 Appendix A. Graphics with Linear Scales

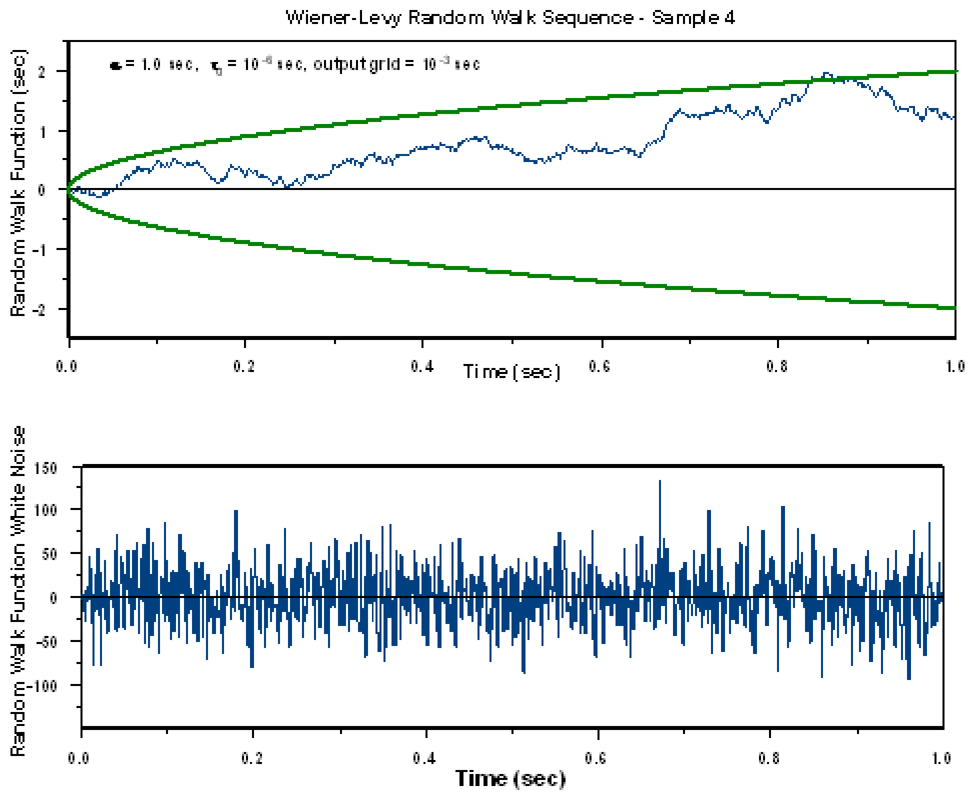

Figure 2: Random Walk and White Noise

4 Appendix B. Derivation of Allan Root-Variance Diagram Slopes

Define graphics variables:

$$\eta = \log_{10} \tau, \qquad \tau = 10^{\eta}$$

$$f\left(\eta\right) = \log_{10} \sigma_y\left(\eta\right), \qquad \sigma_y\left(\eta\right) = 10^{f(\eta)}$$

Differentiate $f\left(\eta\right)$:

$$\frac{df\left(\eta\right)}{d\eta} = \frac{1}{\left(\log_e 10\right)\left(\sigma_y\left(\eta\right)\right)} \frac{d\sigma_y\left(\eta\right)}{d\eta}$$

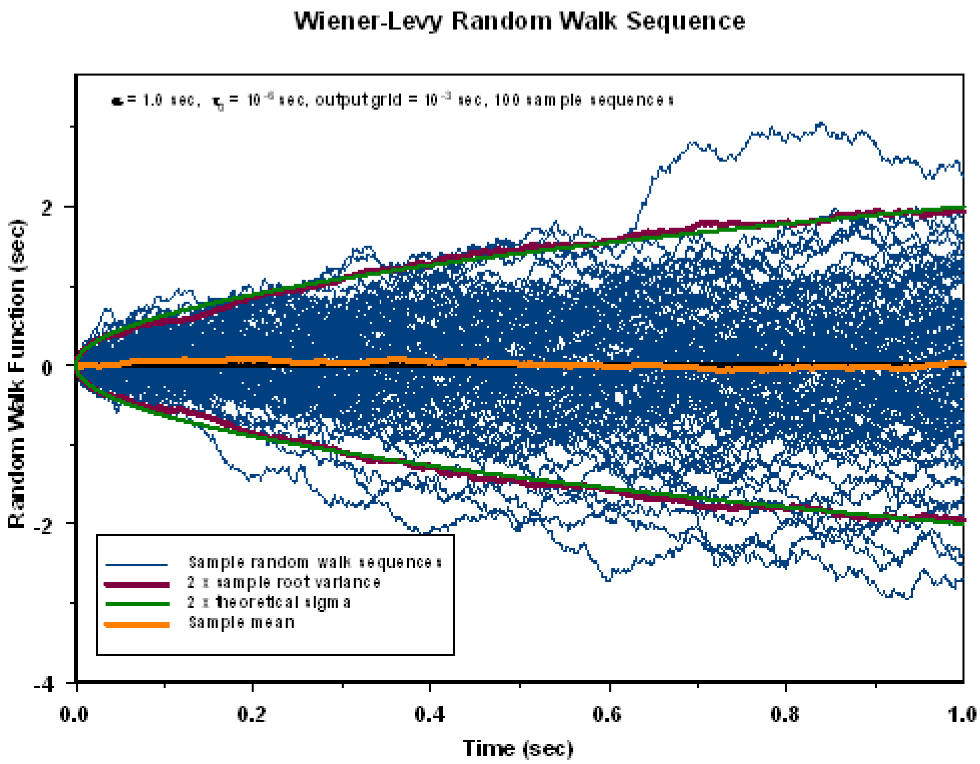

Figure 3: Random Walk

4.1 White Frequency Noise

$$\sigma_y(\eta) = \sqrt{a_0} 10^{-\eta/2}$$

$$\log_{10} \sigma_y(\eta) = \log_{10} \sqrt{a_0} 10^{-\eta/2}$$

That is:

$$f(\eta) = \log_{10} \sqrt{a_0} 10^{-\eta/2}$$

Then:

$$\frac{df(\eta)}{d\eta} = \frac{1}{(\log_e 10)\left(\sqrt{a_0}10^{-\eta/2}\right)} \frac{d\left(\sqrt{a_0}10^{-\eta/2}\right)}{d\eta}$$

where:

$$\frac{d\left(\sqrt{a_0}10^{-\eta/2}\right)}{d\eta} = (-1/2)(\log_e 10)\left(\sqrt{a_0}10^{-\eta/2}\right)$$

Then:

$$\frac{df(\eta)}{d\eta} = -1/2$$

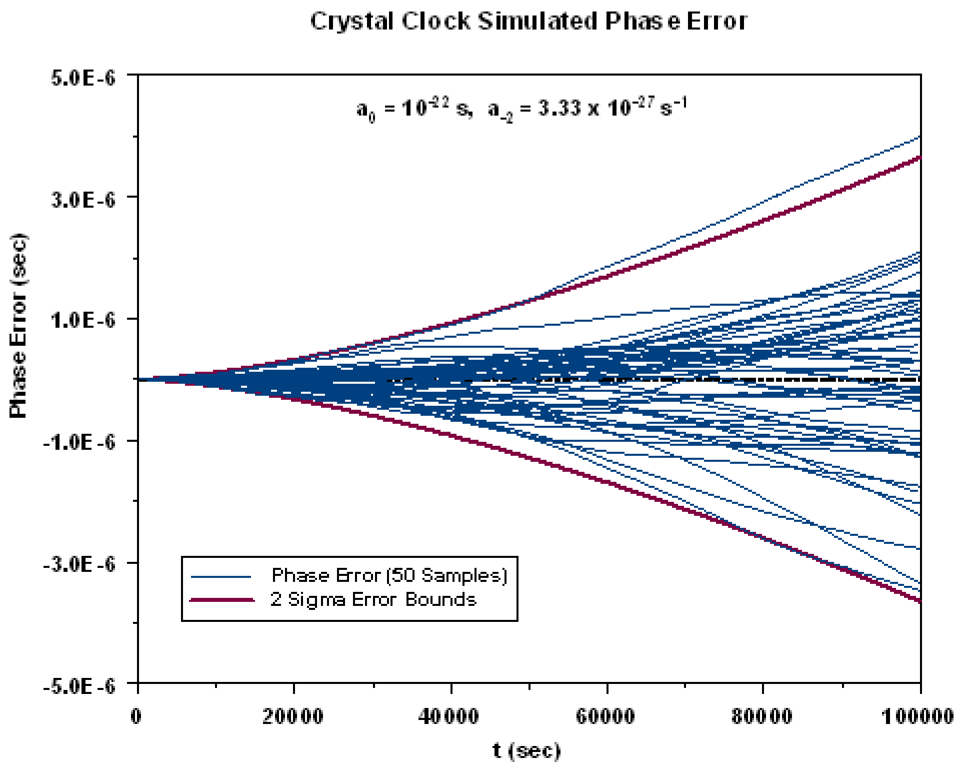

Figure 4: Crystal Clock Phase Error

4.2 Random Walk

$$\sigma_y(\eta) = \sqrt{a_{-2}}10^{\eta/2}$$

$$\log_{10}\sigma_y(\eta) = \log_{10}\sqrt{a_{-2}}10^{\eta/2}$$

That is:

$$f(\eta) = \log_{10}\sqrt{a_{-2}}10^{\eta/2}$$

Then:

$$\frac{df(\eta)}{d\eta} = \frac{1}{(\log_e 10)\left(\sqrt{a_{-2}}10^{\eta/2}\right)}\frac{d\left(\sqrt{a_{-2}}10^{\eta/2}\right)}{d\eta}$$

where:

$$\frac{d\left(\sqrt{a_{-2}}10^{\eta/2}\right)}{d\eta} = (1/2)(\log_e 10)\left(\sqrt{a_{-2}}10^{\eta/2}\right)$$

Then:

$$\frac{df(\eta)}{d\eta} = 1/2$$

References

[1] Allan, D.W., *Time and Frequency Characterization, Estimation, and Prediction of Precision Clocks and Oscillators*, NIST Technical Note 1337, 1990

[2] Chuba, William A., Creation of Graphics at AGI

[3] Wright, James R., Chuba, William A., *Stochastic Clock Model for Definitive Sequential Orbit Determination*, AAS/AIAA Astrodynamics Specialist Conference, Halifax, 14-17 Aug., 1995 (Note: The last half of Page 7 beginning with "Then for ...", and first line (Eq. 3.17) of Page 8, should be stricken)