Tracking System Calibration Validation  
with  
Orbit Determination Tool Kit (ODTK)  

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1 Introduction

1.1 Purpose

My primary goal here is to validate, with real data, the correct response to measurement residual input user controls for ODTK\(^1\).

1.2 Secondary Goals

- To identify white noise in range and Doppler measurement residuals
- To provide a tutorial for tracking-system calibration using ODTK.
- To demonstrate optimal ODTK performance on real tracking data.

1.3 Conclusions

- Measurement residual input user controls for ODTK have been validated with real data. Most dramatic, the correct response to variations of input calibration controls for range statistics has been demonstrated by comparison of range histograms for Runs 1 and 22. This supports my primary validation goal.

- The position and velocity error root-variances are reduced due to reducing the range residual ratio denominators (range residual root-variances) via user input calibration controls. This supports my primary validation goal.

\(^1\)This document was written on 13 May 2002, and then modified on 02 Sep 2004 to change name STK/OD to ODTK.
• Band-limited Gaussian white noise is visually apparent in range histograms and most graphs of range residuals for Run 22. Operationally this is desired. This supports primary and secondary validation goals.

• The absence of band-limited white noise has been clearly demonstrated in all Doppler histograms and in some graphs ("butterfly" patterns) of Doppler residuals for Run 22. Operationally this is not desired. This questions the current numerical method of differencing in the calculation of range differences for Doppler representations, questions the differencing for Doppler residuals, and questions sufficiency in the word length for sensor produced Doppler values.

• Input calibration controls were used to demonstrate that band-limited white noise could not be achieved for Doppler residuals. This supports my primary validation goal.

• Variations in ODTK performance with real AFSCN (.cob) tracking data has been demonstrated. Optimal performance cannot be asserted until the source of obscuration of thermal noise in the Doppler measurement residuals has been determined. See the discussion of Obscured Thermal Noise on Doppler Residuals below.

• A partial tutorial for tracking system calibration using ODTK has been provided. This supports a secondary goal.

2 The Tracking Data

In December of 2000, AGI did request, from Mr Tom Gidlund (Lockheed-Martin), extensive AFSCF real tracking data for an Air Force LEO spacecraft subject to significant air-drag. In January 2001 Tom collected and delivered to AGI (with Air Force approval) range, Doppler, azimuth, and elevation data for vehicles V3333 and V6666. Both spacecraft were equipped with transponders. The tracking data span for both vehicles began on 1 July 2000, and terminated on the last day of September 2000. The tracking data for this time span did include the effects of a violent CME (solar coronal mass ejection) at solar maximum with significant Earth impact. According to Tom, V3333 presented one of the most difficult cases for least squares orbit determination in his operational career (35 years).

3 Tracking System Description

In essence, any tracking system consists of ground stations and space objects. Each ground station contains a clock-driven transmitter, or transmitter/receiver, and each space object may or may not contain a receiver and/or a transmitter, or a clockless receiver/transmitter (transponder). Electromagnetic plane waves are emitted by one or more transmitters and are sensed by one or more
receivers. Many tracking systems transmit at radio frequencies. In this case, a transmitter/receiver is referred to as a radar. Passive space objects reflect electromagnetic plane waves, at significantly reduced power, and can be tracked by a receiver designed for this purpose.

3.1 Range

3.1.1 Range Measurement Definition

The range measurement is the time of receipt of a plane wave by a receiver, less the time of emission of the same plane wave by a transmitter. Thus the range measurement is the time delay of plane wave from transmitter to receiver. Physically, the range measurement time delay is composed of a sum of terms. One term is associated with the finite constant speed of any electromagnetic wave in vaccuo. Another term is associated, when appropriate, with time delay due to tropospheric refraction, and another term is associated with time delay due to ionospheric refraction. If two distinct clocks are used to measure time delay, one clock in transmitter and another in receiver, then there is time delay associated with related clock phenomenology, and time delay associated with general and special relativity. Ground station cables and spacecraft cables used in the transmission of signals contribute time delay terms. Antenna electromagnetic phase center variations contribute time delay terms, and rotation of transmitter antenna with respect to receiver antenna contributes an antenna polarization induced time delay (Faraday rotation).

The range measurement is typically converted from time units to distance units by multiplication with the speed of light constant.

3.1.2 Range Measurement Representation

A range measurement representation is calculated by ODTK, using a state estimate, to model the time delay. The state estimate has estimation errors. The representation is subtracted from the measurement to form a measurement residual. The representation is subject to many types of modeling errors, hence the measurement residual is always erroneous.

3.1.3 Range Residual Modeling Errors

The collection of range residual modeling errors is partitioned into classes associated with range measurement definition. Each term in the range definition must be modeled in the representation, and each model is erroneous.

3.2 Doppler

3.2.1 Doppler Measurement Definition

The Doppler measurement is a phase count (an integer, or integer plus a fraction) on the radio signal carrier across an arbitrary time interval defined by the receiver clock.
3.2.2 Doppler Measurement Representation

A Doppler measurement representation is calculated by ODTK, using a state estimate, to model the phase count. The state estimate has estimation errors. The Doppler measurement representation can be written essentially as the difference of two range representations (the Doppler Representation Theorem), one at the final time of the phase count interval, and one at the initial time of the phase count interval. The Doppler measurement representation is subtracted from the measurement to form a Doppler measurement residual. The Doppler representation is subject to the same modeling errors as for range (i.e., range difference), hence the Doppler measurement residual is always erroneous.

3.2.3 Doppler Residual Modeling Errors

The collection of Doppler residual modeling errors is partitioned into classes associated with Doppler measurement definition. Each term in the Doppler definition must be modeled in the representation, and each model is erroneous.

4 Range and Doppler Statistics

Input controls for the tables below on range and Doppler statistics are denoted and defined:

- \( c_{bias} \) time constant bias, independent of Gauss-Markov bias
- \( \sigma_{GM\ bias} \) sigma on white noise sequence that drives Gauss-Markov bias sequence
- \( \tau \) exponential half-life on Gauss-Markov bias sequence
- \( \sigma_{WN\ bias} \) sigma on thermal white noise, independent of Gauss-Markov bias

Do not confuse the distinct sigmas for white noise. Input constant \( \sigma_{GM\ bias} \) defines white noise root-variance that controls, in part, the Gauss-Markov bias sequence model. Input constant \( \sigma_{WN\ bias} \) defines completely the thermal white noise root-variance control, and is independent of the Gauss-Markov model.

We will not address the constant bias \( c_{bias} \) and thermal noise sigma \( \sigma_{WN\ bias} \) further in this document.

4.1 A Priori Range and Doppler Bias Models

First consider range and Doppler bias models in the absence of range and Doppler measurements. Model Eq. 1 defines the abstract basis on which to build an estimation capability. Range and Doppler biases are modeled in ODTK as separate time-varying Gauss-Markov sequences with common stochastic structure. Let \( \rho \) denote range, and let \( \psi \) denote Doppler. Let \( \beta_\rho(t) \) denote range bias with time, and let \( \beta_\psi(t) \) denote Doppler bias with time.
Since both bias sequences have the same structure, let $x (t)$ denote either; that is: $x (t) \in \{ \beta_p (t), \beta_p (t) \}$.

Let $x = x (t_k)$ denote a dynamic scalar random variable that satisfies the equation:

$$x (t_{k+1}) = \Phi (t_{k+1}, t_k) x (t_k) + \sqrt{1 - \Phi^2 (t_{k+1}, t_k)} w (t_k), \quad k \in \{0, 1, 2, \ldots \} \tag{1}$$

where $w (t)$ is a Gaussian white random variable with mean zero and constant variance $\sigma^2_w$, and where:

$$x (t_0) = w (t_0) \tag{2}$$

$$\Phi (t_{k+1}, t_k) = e^{\alpha |t_{k+1} - t_k|} \tag{3}$$

constant $\alpha < 0 \tag{4}$

Let $E \{ \}$ denote the linear expectation operator, and define:

$$\sigma^2_{GM \ bias} = E \{ w^2 (t_0) \} \tag{5}$$

Then with Eq. 2:

$$E \{ x^2 (t_k) \} = \sigma^2_{GM \ bias} \tag{6}$$

It is demonstrated below that:

$$E \{ x^2 (t_k) \} = \sigma^2_{GM \ bias}, \text{ constant for each } k \tag{7}$$

Since the variance $E \{ x^2 (t_k) \}$ has the same value for any $t_k$, then $x (t_k)$ is said to be a stationary sequence. But note: Eq. 1 shows that $x (t_k)$ is not constant with time.

**4.1.1 Stationary Variance**

For $k = 0$, Eq. 1 becomes:

$$x (t_1) = \Phi (t_1, t_0) x (t_0) + \sqrt{1 - \Phi^2 (t_1, t_0)} w (t_0) \tag{8}$$

Then use this and Eq. 6 to get:

$$E \{ x^2 (t_1) \} = (\Phi (t_1, t_0))^2 E \{ x^2 (t_0) \} + (1 - \Phi^2 (t_1, t_0)) E \{ w^2 (t_0) \} = \sigma^2_{GM \ bias} \tag{9}$$

because:

$$E \{ x (t_0) w (t_0) \} = 0 \tag{10}$$

Eq. 7 follows by induction on the integers $k \in \{0, 1, 2, \ldots \}$. 7
4.1.2 Propagation Time Extrema
The stochastic sequence defined by Eq. 1 depends on $|t_{k+1} - t_k|$ as follows.

**Unity Serial Correlation**

$$(|t_{k+1} - t_k| = 0) \implies (\Phi (t_{k+1}, t_k) = 1), \text{ and } (x(t_{k+1}) = x(t_k)) \quad (11)$$

**White Noise**

$$(|t_{k+1} - t_k| = \infty) \implies (\Phi (t_{k+1}, t_k) = 0), \text{ and } (x(t_{k+1}) = w(t_k)) \quad (12)$$

4.1.3 User Input Control On Serial Correlation
Refer to Eq. 3, and replace $|t_{j+1} - t_j|$ with $|\tau|$. It is convenient to set the value for $\alpha$ by choosing a constant value of $\tau$ associated with exponential half-life on the transition function $\Phi$:

$$(e^{\alpha \tau} = 0.5) \implies (\alpha = (\ln 0.5)/\tau) \quad (13)$$

Equation 13 converts the exponential half life user input control $\tau$ to $\alpha$, the driving constant for $\Phi$.

**Serial Correlation** Theoretically:

$$(\tau = \infty) \implies (\alpha = 0) \implies (\Phi (t_{k+1}, t_k) = 1), \text{ and } (x(t_{k+1}) = x(t_k)) \quad (14)$$

Practically, the user forces the sequence in $x$ to be more correlated by setting $|\tau| < \infty$ larger.

**White Noise** Theoretically, to guarantee white noise (zero serial correlation):

$$(\tau = 0) \implies (\alpha = -\infty) \implies (\Phi (t_{k+1}, t_k) = 0), \text{ and } (x(t_{k+1}) = w(t_k)) \quad (15)$$

Practically, the user forces the sequence in $x$ to be less correlated by setting $|\tau| > 0$ smaller.

4.1.4 User Input Control On White Noise Variance
The user specifies the sigma $\sigma_{GM\ bias}$ on $w(t_0)$. Practically, one matches $\sigma_{GM\ bias}$ to the physical sigma on $x(t)$, in the absence of measurement information.
4.2 Filter Time Update Equations for Range and Doppler Biases

It is convenient here to convert time-tag notation to subscripts; i.e., \( \Phi_{k+1,k} \equiv \Phi(t_{k+1}, t_k), \hat{x}_{i|h} \equiv \hat{x}(t_i|t_h), \) etc.

4.2.1 Bias Estimates

Let \( y_j \) denote either a range or a Doppler measurement at time \( t_j \). Now consider range and Doppler bias models for use with range and Doppler measurements. Given the bias estimate \( \hat{x}_{k+j} \) at time \( t_k \), due to previous processing of measurement \( y_k \) at time \( t_k \), and given measurement \( y_{k+1} \) at time \( t_{k+1} \), then the optimal estimate \( \hat{x}_{k+1+j} \) of the true unknown bias \( x_{k+1} \) at time \( t_{k+1} \) is defined, in part, by Sherman’s Theorem to be the conditional mean:

\[
\hat{x}_{k+1} = E\{x_{k+1}|y_{k+1}\} \tag{16}
\]

Run the expectation operator through Eq. 1 to get:

\[
\hat{x}_{k+1} = \Phi_{k+1,k}\hat{x}_{k|k} \tag{17}
\]

where:

\[
E\{w_{k+1}|y_{k+1}\} = 0 \tag{18}
\]

Eq. 17 defines the filter Time Update function to move the known estimate \( \hat{x}_{k|k} \) (derived from the filter Measurement Update) from time \( t_k \) to time \( t_{k+1} \) to obtain \( \hat{x}_{k+1|k} \). Note that Eq. 17 drives the estimate \( \hat{x}_{k|k} \) toward zero. The rate of return to zero depends on the user input value for half-life \( \tau \).

4.2.2 Bias Estimate Errors

For any \( i, j, k \in \{0, 1, 2, \ldots\} \), define the unknown error \( \delta\hat{x}_{i|h} \) in \( \hat{x}_{i|h} \) with:

\[
\delta\hat{x}_{i|j} = x_i - \hat{x}_{i|h} \tag{19}
\]

Then the unknown error \( \delta\hat{x}_{k+1|k} \) in \( \hat{x}_{k+1|k} \) is:

\[
\delta\hat{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} \tag{20}
\]

Insert Eqs. 1 and 17 into Eq. 20 to get:

\[
\delta\hat{x}_{k+1|k} = \Phi_{k+1,k}\delta\hat{x}_{k|k} + \sqrt{1 - \Phi_{k+1,k}^2} w_k \tag{21}
\]

where \( \delta\hat{x}_{k|k} \) is the unknown error in \( \hat{x}_{k|k} \). Note that Eqs. 1 and 21 have the same structure. In fact, the second term on the right is identical for both equations. But these equations are very different because Eq. 1 models the unknown bias and Eq. 21 models the unknown error in the known estimate of the bias. The variance on the error \( \delta\hat{x}_{k+1|k} \) is derived:
This is the applied filter Time Update equation for bias estimate error variances. It demonstrates that $\delta x_{k+1|k}$ is not a stationary sequence; i.e., variances are not time constants, not equivalent to Eq. 7. The filter Measurement Update has previously reduced the propagated bias error variance $E\left\{\delta x^2_{k|k}\right\}$ to $E\left\{\delta x^2_{k|k-1}\right\}$, due to processing measurement $y_k$ at time $t_k$. With $k \in \{0, 1, 2, \ldots\}$, the recursive processing of densely spaced measurements invokes both the filter Measurement Update and filter Time Update (Eq. 22), and these measurements drive the bias error variances $E\left\{\delta x^2_{k|k}\right\}$ and $E\left\{\delta x^2_{k+1|k}\right\}$ toward zero. But in the absence of measurements, $E\left\{\delta x^2_{k+1|k}\right\}$ returns to $\sigma_{GM}^2$ bias. The rate of return depends on the user input value for half-life $\tau$.

4.2.3 User Input Control

Eq. 22 is controlled by two user inputs. The first is the root-variance $\sigma_{GM}^2$ bias on the Gauss-Markov band-limited white noise. And the second is the exponential half-life $\tau$. Equation 13 converts the user input control $\tau$ to $\alpha$, the driving constant for $\Phi_{k+1,k}$. The behaviour of $\tau$ on $\Phi_{k+1,k}$ is explained with Eqs. 14, and 15.

5 Histogram Graph

All histogram graphs herein are derived from Runs 1 and 22, in which the same 26 days of real measurement data were processed for V3333.

The purpose for construction of each histogram graph of measurement residual ratios is to compare it to the Gaussian density function $f(x)$, then to modify histogram input controls so as to visually derive, if possible, a histogram that approximates the Gaussian density function.

Why would one construct such a comparison?

For range and Doppler electronic measurements, the physical lower bound on measurement residual root-variance is derived from Gaussian distributed white noise – thermal noise. And Gauss himself used Gaussian white noise to model white noise on angles measurement residuals for orbit determination. If Gaussian white noise can be identified for measurement residual ratios, then an important necessary condition is demonstrated for optimal filter performance. And if, in the process of modifying histogram input controls, the measurement residual root-variance ratio denominators are reduced, then estimated spacecraft orbit error root variances will also be reduced. If ratio denominators are increased, then estimated spacecraft orbit error root variances will also be increased. Thus, in part, a realistic orbit error covariance function is generated.

If Gaussian white noise cannot be identified for measurement residual ratios, then the physical lower bound on measurement residuals is shown to be masked
by other effects. Examples: (i) Absence of an appropriate measurement bias model for the state estimate structure, or defects in the measurement bias model used; (ii) Sensor hardware truncation and round-off of its output measurement values, or loss of double precision mantissa significance due to differencing of similar measurement and/or measurement representation values. In the former case, the measurement bias model problem should be solved. In the latter case, an important software and algorithm diagnostic is defined, and should be explained and fixed.

The Central Limit Theorem (Appendix A) explains, in principle, why Gaussian histograms could be found even when the physical lower bound on measurement residuals has not been identified.

5.1 Gaussian Functions

For any Gaussian distribution, let \( \mu \) and \( \sigma \) denote its mean and root-variance. Then the scalar Gaussian (or normal\(^2\)) distribution function \( F(x) \) for \( (\mu, \sigma) = (0, 1) \) is defined by:

\[
F(x) = \int_{-\infty}^{x} f(\eta) d\eta = 0.5 + \int_{0}^{x} f(\eta) d\eta
\]  

(23)

where:

\[
f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)
\]  

(24)

Function \( f(x) = dF(x)/dx \) is referred to as the Gaussian (or normal) density function.

![](image1.png)

Figure 1: Gaussian Density Function f(x)

\(^{2}\) According to Feller, the reference to Gauss for normal distribution and density functions is historically inaccurate because they were used earlier by DeMoivre and LaPlace in probability theory.
5.2 Histogram Construction

5.2.1 An Ideal Histogram

Let \( \Delta x = x_{k+1} - x_k > 0 \) where \( k \) is an integer, define a constant histogram bin width. And let \( A(x_k, x_{k+1}) \) denote the area for bin \((x_k, x_{k+1})\). Then:

\[
A(x_k, x_{k+1}) = F(x_{k+1}) - F(x_k)
\]  \hspace{1cm} (25)

Then:

\[
A(x_k, x_{k+1}) = \int_{x_k}^{x_{k+1}} f(y) \, dy
\]  \hspace{1cm} (26)

The bin height is defined by:

\[
h(x_k, x_{k+1}) = A(x_k, x_{k+1}) / \Delta x
\]  \hspace{1cm} (27)

Compare \( h(x_k, x_{k+1}) \) to \( f((x_k + x_{k+1})/2) \) on the graph. It will not be an exact match because \( \Delta x > 0 \).

Example Given \( x_k = 0, \) \( x_{k+1} = 0.1 \). Then \( \Delta x = 0.1, A(x_k, x_{k+1}) \approx 0.03982784, h(x_k, x_{k+1}) \approx 0.3982784, \) and \( f((x_k + x_{k+1})/2) \approx 0.3984439.\)

5.2.2 Real Histograms

Given sample size \( N \), with \( M_{[x_k, x_{k+1}]} \) samples in bin \([x_k, x_{k+1}]\) of width \( W_{[x_k, x_{k+1}]} = x_{k+1} - x_k \), find ratio:

\[
R_{[x_k, x_{k+1}]} = \frac{M_{[x_k, x_{k+1}]}}{N}
\]

Interpret \( R_{[x_k, x_{k+1}]} \) as the bin area \( A_{[x_k, x_{k+1}]} = R_{[x_k, x_{k+1}]} \). Divide the bin area \( A_{[x_k, x_{k+1}]} \) by the bin width \( W \) to get the bin height \( H_{[x_k, x_{k+1}]} \) at center of bin \((x_{k+1} + x_k)/2\):

\[
H_{[x_k, x_{k+1}]} = \frac{A_{[x_k, x_{k+1}]} / W}{N}
\]

We multiply \( H_{[x_k, x_{k+1}]} \) by 100 to provide results in terms of density percentages.

Example Suppose \( N = 1.0 \times 10^6 \) for bin \([x_k, x_{k+1}] = [0.0, 0.1]\), and \( M_{[0.0,0.10]} = 39844. \) Then \( A = R = 0.039844 \) and \( H = A / W = 0.398444 \). Finally \( H \times 100 = 39.8444\%\).

5.2.3 Sample Size

When sample size \( N \) is small, even a histogram of Gaussian samples will not visually approximate a Gaussian density function. By small: \( N < 1000 \). When sample size \( N \) is large, a histogram of Gaussian samples will visually approximate a Gaussian density function very well. By large: \( N > 2 \times 10^6 \). Our largest population sizes of real samples are near 4000 and 5000.
6 Range Residual Ratio Histograms

Calibration runs were performed, where each run processed range, Doppler, azimuth, and elevation measurements, for V3333 from 1 July 2000 through 26 July 2000 (390 orbit periods). Range residual measurement statistics used for operational least squares orbit determination were obtained from Tom Gidlund. These were used to define input user controls for ODTK Run 1. After each run, station dependent range residual ratio histograms were constructed for this twenty-six day span. Stations BOSS-A and HULA-B, with the largest population size, are considered in this report.

6.1 Run 1

It is important to note with least squares that one is severely limited in ability to determine optimal residual measurement statistics values.

6.1.1 BOSS-A

The BOSS-A histogram for Run 1 is biased positive, and is bunched and peaked relative to the unbiased Gaussian density function. The positive histogram bias can be removed by increasing the value of the stored constant bias$^3$.

$^3$The stored time constant bias is not to be confused with the time varying Gauss-Markov (GM) "bias" estimate. The latter always returns to zero exponentially in the absence of measurements, and thus must be re-estimated with more measurements. The rate at which...
Figure 3: Range Histogram HULA-B Run 1

<table>
<thead>
<tr>
<th>Station</th>
<th>c bias (m)</th>
<th>$\sigma_{GM}$ bias (m)</th>
<th>$\tau$ (min)</th>
<th>$\sigma_W$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BOSS A</td>
<td>563</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>BOSS B</td>
<td>654</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>COOK A</td>
<td>493.25</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>COOK B</td>
<td>528</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>GUAM A</td>
<td>252</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>GUAM B</td>
<td>278</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>HULA A</td>
<td>242</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>HULA B</td>
<td>264</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>LION A</td>
<td>355</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>PIKE A</td>
<td>211</td>
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<td>5</td>
<td>7.62</td>
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<td>100</td>
<td>5</td>
<td>7.62</td>
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<td>POGO B</td>
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<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>POGO C</td>
<td>301</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
<tr>
<td>REEF A</td>
<td>193</td>
<td>100</td>
<td>5</td>
<td>7.62</td>
</tr>
</tbody>
</table>

Table 1: Range Statistics for Run 1
The histogram can be spread by reducing the size of the ratio denominator; i.e., by reducing the measurement residual root-variance (sigma). This can be practically achieved in two ways: (i) Reduce the range thermal noise variance, (ii) Reduce the estimated Gauss-Markov range bias root-variance.

6.1.2 HULA-B

The HULA-B histogram for Run 1 is bunched and peaked relative to the unbiased Gaussian density function. The histogram can be spread by reducing the size of the ratio denominator; i.e., by reducing the measurement residual root-variance (sigma). This can be practically achieved in two ways: (i) Reduce the range thermal noise variance, (ii) Reduce the estimated Gauss-Markov range bias root-variance.

6.2 Run 22

6.2.1 BOSS-A

The Gauss-Markov range residual bias sigma for Run 1 (100m) was reduced to 30m for Run 22, and the range residual white noise sigma for Run 1 (7.62m) was reduce to 1.00m for Run 22. Comparison of range residual ratio histograms, for Run 1 and Run 22, exhibits a transformation from non-Gaussian to Gaussian sample density function – approximately. This supports the validation. Also, it is probable that Gaussian thermal noise is visible.

\[
\text{the GM bias returns to zero depends on the GM half-life } \tau \text{ assigned by the user. When it is clear to the user that there exists a non-zero mean GM bias, then that mean bias value should be identified and entered by the user as a constant bias. Then remaining bias magnitudes will be smaller. This enables the filter to respond with GM bias estimates to bias time variations more quickly, and can significantly improve filter responsiveness.}
\]
6.2.2 HULA-B

The Gauss-Markov range residual bias sigma for Run 1 (100m) was reduced to 28m for Run 22, and the range residual white noise sigma for Run 1 (7.62m) was reduced to 1.00m for Run 22. Comparison of range residual ratio histograms, for Run 1 and Run 22, exhibits a transformation from non-Gaussian to Gaussian sample density function – approximately. This supports the validation. Also, it is probable that Gaussian thermal noise is visible.

7 Doppler Residual Ratio Histograms

Figure 5: Range Histogram HULA-B Run 22

Figure 6: Doppler Histogram BOSS-A Runs 1 & 22
<table>
<thead>
<tr>
<th></th>
<th>( c ) bias (m)</th>
<th>( \sigma_{\text{GM bias}} ) (m)</th>
<th>( \tau ) (min)</th>
<th>( \sigma_{\text{WN}} ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BOSS A</td>
<td>800</td>
<td>30</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>BOSS B</td>
<td>1015</td>
<td>30</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>COOK A</td>
<td>765</td>
<td>32</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>COOK B</td>
<td>793</td>
<td>30</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>GUAM A</td>
<td>190</td>
<td>30</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>GUAM B</td>
<td>225</td>
<td>38</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>HULA A</td>
<td>135</td>
<td>28</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>HULA B</td>
<td>210</td>
<td>28</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>LION A</td>
<td>505</td>
<td>44</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>PIKE A</td>
<td>80</td>
<td>25</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>POGO A</td>
<td>340</td>
<td>32</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>POGO B</td>
<td>230</td>
<td>40</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>POGO C</td>
<td>320</td>
<td>28</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>REEF A</td>
<td>48</td>
<td>32</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Range Statistics for Run 22

<table>
<thead>
<tr>
<th></th>
<th>( c ) bias (cm/s)</th>
<th>( \sigma_{\text{GM bias}} ) (cm/s)</th>
<th>( \tau ) (min)</th>
<th>( \sigma_{\text{WN}} ) (cm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BOSS A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>BOSS B</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>COOK A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>COOK B</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>GUAM A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>GUAM B</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>HULA A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>HULA B</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>LION A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>PIKE A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>POGO A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>POGO B</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>POGO C</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>REEF A</td>
<td>0</td>
<td>100</td>
<td>20</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 3: Doppler Statistics for Runs 1 and 22
Calibration runs were performed, where each run processed range, Doppler, azimuth, and elevation measurements, for V3333 from 1 July 2000 through 26 July 2000 (390 orbit periods). Doppler residual measurement statistics used for ODTK Runs 1 and 22 were defined by Jim Fields. After each run, station dependent Doppler residual ratio histograms were constructed for this twenty-six day span. Stations BOSS-A and HULA-B, with the largest population size, are considered in this report.

Doppler histograms for BOSS-A and HULA-B (Runs 1 and 22) are somewhat peaked and bunched. When ratio denominators are reduced, the associated histograms display symmetric mid-range voids. Variations in input measurement statistics indicate that it is impossible to produce visually acceptable Gaussian histogram approximations. It is therefore assumed that band-limited Gaussian white noise (thermal noise) is obscured in the Doppler measurement residuals.

8 Residual Graphs with Time

All graphs are derived from Run 22, in which 26 days of real measurement data were processed for V3333. The purpose for displaying the graphs herein is to aid in the identification, or negation, of measurement residual band-limited white noise (Note that non-band-limited white noise has infinite variance). With any white noise, each measurement residual at each time \( t \) should be independent of residuals prior to \( t \), and independent of residuals after \( t \). Visually, a white noise graph should not present serial correlation in the residual pattern. With the small ensembles of residuals displayed herein, it is sometimes difficult to identify serial correlation. However, repeating correlated patterns are easy to identify.

Measurement residuals are graphed as red dots with time (Minutes Past Greenwich Midnight). Residual editing boundarys are overlaid in black, symmetrically at \( \pm 3\sigma \). And the estimated Gauss-Markov measurement bias is over-
8.1 Range

8.1.1 BOSS-A

8.1.2 HULA-B

8.2 Doppler

8.2.1 BOSS-A

The Doppler residuals displayed in Figures 13 and 14 for BOSS-A have repeating "butterfly" patterns that demonstrate serial correlation and the absence of thermal noise. The other BOSS-A Doppler residuals Figures displayed also seem to indicate serial correlation.

8.2.2 LION-A

The Doppler residuals displayed in Figure 17 for LION-A has repeating "butterfly" patterns that demonstrate serial correlation and the absence of thermal noise. The other LION-A Doppler residuals Figures displayed also seem to indicate serial correlation.
Figure 9: Range Residuals BOSS-A Run 22 G2

Figure 10: Range Residuals BOSS-A Run 22 G3
Figure 11: Range Residuals HULA-B Run 22 G1

Figure 12: Range Residuals HULA-B Run 22 G2

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Figure 13: Doppler Residuals BOSS-A Run 22 G1

Figure 14: Doppler Residuals BOSS-A Run 22 G2
Figure 15: Doppler Residuals BOSS-A Run 22 G3

Figure 16: Doppler Residuals BOSS-A Run 22 G4
Figure 17: Doppler Residuals LION-A Run 22 G1

Figure 18: Doppler Residuals LION-A Run 22 G2
9 Discussion of Obscured Thermal Noise on Doppler Residuals

The Doppler measurement residuals $\Delta \psi = \psi - \hat{\psi}$ are formed by subtracting Doppler representations $\hat{\psi}$ (measurement estimates) from sensor generated Doppler measurements $\psi$. The obvious places to look for obscuration of the thermal noise on Doppler measurement residuals are in the Doppler measurements $\psi$, the Doppler measurement representations $\hat{\psi}$, and in loss of significance due to the difference operation $\Delta \psi = \psi - \hat{\psi}$.

The sensor generated Doppler measurements $\psi$ are given in integers with units ($m/s \times 10^4$). Thus the rounding error $\delta \psi$ is bounded according to $|\delta \psi| < 0.5 \times 10^{-4}m/s = 0.05mm/s$. Is this sufficiently large to obscure thermal noise?

The Doppler measurement representations $\hat{\psi}$ are formed by ODTK in double precision as the ratio $\hat{\psi} = \Delta \hat{\rho} / \Delta t$ of range difference $\Delta \hat{\rho}$ divided by the time difference $\Delta t$, where $\Delta \hat{\rho}$ is the change in estimated range that occurs across time $\Delta t$. If the time $\Delta t = t_{k+1} - t_k$ is short, then $|\Delta \hat{\rho}|$ may be very small. Is it small enough to lose significance in the difference $\Delta \hat{\rho}_{k+1,k} = \hat{\rho}(t_{k+1}) - \hat{\rho}(t_k)$? If so, is the representation error magnitude $|\delta \hat{\psi}|$ sufficiently large to obscure thermal noise?
Table 4: Compare Orbit Elements Run 1 vs Run 22

<table>
<thead>
<tr>
<th>Run</th>
<th>a (km)</th>
<th>e</th>
<th>u (deg)</th>
<th>i (deg)</th>
<th>Omega (deg)</th>
<th>omega (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 1</td>
<td>6879.744697</td>
<td>0.002161</td>
<td>27.189097</td>
<td>105.025980</td>
<td>41.995028</td>
<td>125.380103</td>
</tr>
<tr>
<td>Run 22</td>
<td>6879.744353</td>
<td>0.002161</td>
<td>27.189205</td>
<td>105.025984</td>
<td>41.995067</td>
<td>125.354077</td>
</tr>
</tbody>
</table>

Table 5: Orbit Error Sigmas (RIC) Run 1 vs Run 22

<table>
<thead>
<tr>
<th>Run</th>
<th>(\sigma_R) (m)</th>
<th>(\sigma_I) (m)</th>
<th>(\sigma_C) (m)</th>
<th>(\sigma_R) (cm/s)</th>
<th>(\sigma_I) (cm/s)</th>
<th>(\sigma_C) (cm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 1</td>
<td>5.3</td>
<td>21.3</td>
<td>7.5</td>
<td>2.4</td>
<td>5.2</td>
<td>8.9</td>
</tr>
<tr>
<td>Run 22</td>
<td>4.4</td>
<td>11.5</td>
<td>5.6</td>
<td>1.3</td>
<td>4.3</td>
<td>6.8</td>
</tr>
</tbody>
</table>

10 Orbit Comparisons

Orbit values are given at the time of last measurement processed: 26 July 2000 21h 11m 33s.

10.1 Kepler Orbit Elements

See Table 4 for inputs to the following calculation. The difference in \(u\) : \(\Delta u = 1 \times 10^{-4} \text{deg} = 1.74 \times 10^{-6} \text{rad}\). The difference in in-track positions: \(\Delta I = (\Delta u) (a) = 12m\). Compare this to \(\sigma_I\) (m) for Runs 1 and 22 (Table 5). The difference in \(u\) is consistent with the covariance matrices for both Runs.

10.2 Position and Velocity RIC Sigmas

Table 5 demonstrates that the position and velocity error root-variances are reduced due to reducing the range residual ratio denominators (root-variances). This supports the validation.

A Appendix. Central Limit Theorem

Let random variable \(x\) be the bounded sum of many small independent components with zero means, but where each component may derive from any distribution. Then \(x\) will have a Gaussian distribution function, whose derivative is a Gaussian density function \(f_x(x)\), defined by Eq. 24. See Feller [1] for proof.

References
