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TECHNICAL IMPLEMENTATION OF THE CIRCULAR RESTRICTED THREE-BODY MODEL IN STK ASTROGATOR

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Spacecraft trajectory design is captivating: it is challenging, intriguing, creative, occasionally tedious, and ultimately pretty cool. While some of this characterization is subjective, it is language that is typical for practitioners of the associated arts. One thing that is much more objective is that trajectory design is a process, and this process is often iterative. Currently, the process frequently begins with lower-fidelity, yet representative, models to compute initial guess data for phases of the trajectory itinerary, with incrementally more complex models expanding understanding of the design space. An example of a model that inherently reflects such an incremental increase in fidelity is the Circular Restricted Three-body Problem (CR3BP), which still represents a simplification from real-world systems but does so at the advantage of rich mathematical theory. This model offers reduced numerical and conceptual complexity yet yields solutions that can capture the governing behaviors of the underlying higher-fidelity systems. This paper is intended to establish the relevance of the incorporation of the mathematical framework of the CR3BP model into the Systems Tool Kit (STK) Astrogator module from Analytical Graphics, Inc. (AGI), to discuss the unique implementation requirements posed by this effort and to offer verification and validation of this implementation.

INTRODUCTION

The modeling environment in STK is formulated to incorporate highly accurate systems of space and time. High-fidelity ephemerides are used to define the positions and orientations of solar system bodies. Additional theory, such as high-fidelity modeling of central body orientations and gravity fields, is incorporated to best simulate the dynamical environment where a user's assets are to be modeled. In particular cases, users can even include corrections for general relativity and the Yarkovsky effect. Despite a potentially effective ability to model the system, real solutions deviate from expectations as a consequence of imperfect knowledge of the environment or the spacecraft performance. For example, in a recent interview, the lead scientist for OSIRIS-REx described how outgassing from water ice was substantial enough to yield accelerations larger than solar radiation pressure on the spacecraft and push the vehicle out of orbit about Bennu, had it not been appropriately mitigated.[1,2] While it's ultimately desirable to bring design solutions as close to the real-world solution as reasonably possible, it is often cumbersome, prohibitive and potentially untenable to perform initial design efforts in such high-fidelity environments. For instance, in multi-body trajectory design, lower-fidelity models such as the CR3BP are often employed for rapid analysis and design;

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solutions designed in these approximate models are then efficiently transitioned into higher-fidelity environments for further analysis. To support a trajectory designer who wishes to start from a simplified model, but to also work within the STK system, implementation of the CR3BP model requires special considerations as well as adaptations of the existing environment.

To support efficient transitions between analyses performed in the restricted problem and higher-fidelity environments in STK, a reusable implementation of the CR3BP model is desired. Incorporating simple yet elegant models like the CR3BP into a toolset like Astrogator makes the associated theory a possibility for some mission designers whose focus and time typically lies elsewhere. Bringing the model into the tool also alleviates the previously required user effort associated with transforming data to migrate related analyses into STK as well as post processing. The capability to both create and reproduce CR3BP trajectories directly in Astrogator also enables an additional step in the process of transitioning solutions to higher-fidelity models within the same tool. Beyond these meaningful strides, the current state of a general CR3BP implementation within Astrogator remains somewhat nascent, and the effort represented by this paper reflects a strategy to lay a solid foundation upon which to incrementally expand. Ultimately, if trajectory designers and mission planners can use these tools to do more, faster—to accomplish additional exciting work, then these efforts have been and will continue to be well spent.

This paper supplies an overview of the technical implementation of the lower-fidelity CR3BP within the STK Astrogator framework. First, key concepts associated with STK Astrogator and the circular restricted three-body problem are summarized. Using this foundation, implementation details of the force model within the Systems Tool Kit for use within Astrogator are discussed. Then, specific configuration requirements for the STK system to be used effectively with the CR3BP are described. Examples of direct propagation within STK Astrogator of representative orbits and two L_1 orbit families with initial states taken from external simulation as well as supporting metric comparisons are examined. An L_1 to L_2 transfer produced and corrected in Astrogator is presented and discussed. Finally, concluding remarks are offered along with a few points of future work.

STK ASTROGATOR

Astrogator's roots lie in a strong lineage of tools with incarnations dating back to 1989.[3] Many elements and algorithms preceded these formal offerings, and others have arisen over time. The tool has been used for analysis, design and operations on missions ranging from LEO to GEO,[3] from the Sun[4] to Arrokoth in the Kuiper Belt,[5,6] and many places in between. Most of these applications faced unique requirements, and software enhancements resulted in response.[7–10] Astrogator has also supported numerous programs operationally. The earliest mission utilizing the software was the Wilkinson Microwave Anisotropy Probe (WMAP) mission.[11] In addition to WMAP, Astrogator has also been used in an operational context to support other libration point missions including ARTEMIS (Acceleration, Reconnection, Turbulence and Electrodynamics of the Moon's Interaction with the Sun), a follow-on to THEMIS (Time History of Events and Macroscale Interactions during Substorms) that included phases in and around Earth–Moon L_1 and L_2 and DSCOVR (Deep Space Climate Observatory; Sun–Earth L_1). Many other missions, including various additional libration point missions, have been supported in the planning and analysis phases; some are enumerated in Short et al.[10]

The role of Astrogator in the spacecraft mission design lifecycle is largely reflected in the trajectory design and analysis phases, although it is also relevant to other stages such as early concept design and selection as well as operations. Astrogator's integration within STK leads to significant synergies that enable trade studies where the environment is modeled accurately to capture many important contributions to the system. For example, inherent system awareness of periods of time when a spacecraft has line-of-sight intervisibility (or access) to other resources will affect other aspects of the mission design. Similar access considerations are also critical during operational phases of a spacecraft mission. Further, the integration with STK allows for feedback through graphics and data product reporting.

While the STK system lends some support, Astrogator displays several independent strengths. In turn, Astrogator extends the capabilities of the STK system. Particular strengths of Astrogator are reflected in the principles of modularity and configurability. These principles are implemented in the STK Component Browser and associated component technology.[12] The various components in the browser represent individual aspects of the mission model. A particular component may reflect part of the force model or characteristics associated with a central body for the system. Some components represent calculations and provide a mechanism for reporting various data, while others interact to produce a functional unit or even a sequence of segments to define an Astrogator satellite. These capabilities offer extensive "out-of-the-box" options, while also providing plugin points for users to supply their own models when needed. Additional mechanisms exist to execute the simulation. The Mission Control Sequence (MCS) is a interactive environment for designing and configuring the evolution of spacecraft trajectories. The MCS uses various segments and other objects to capture the design—each of these pieces is a *component*. The recent incorporation of the CR3BP elements that comprise the focus of this paper, while implemented in an integrated way within STK, are all offered to the user through components.

THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

A useful and approximate model of a multi-body system is one that sufficiently captures the dominant features of the dynamical interactions. Formulating a dynamical model that reflects the gravitational interactions of three bodies produces a model that is sufficiently complex to reveal many important characteristics while remaining tractable. However, the general three-body problem possesses no closed-form analytical solution.[13] Thus, additional simplifications, such as those consistent with the Circular Restricted Three-Body Problem*, offer significant insight.

The CR3BP models the gravitational influence of two larger, massive *primaries* (for example, the Earth and the Moon) evolving on circular orbits on a third, much smaller body of negligible mass (e.g., a spacecraft). These two primary bodies are designated as P_1 and P_2. Position variables, x, y, and z describe the position of the third body P_3, the spacecraft, with respect to the barycenter B of the primary system, defined in the rotating frame $(\hat{x}, \hat{y}, \hat{z})$. This rotating frame is depicted in Figure 1 relative to an inertial reference frame $(\hat{X}, \hat{Y}, \hat{Z})$. The system mass parameter is represented by $\mu_{\text{CRP}} = \frac{m_2}{m_1+m_2}$, a function of the masses or mass parameters of the primary bodies. Additionally, distances between the third body and each of the massive primaries are denoted r_{i3}. In a coordinate frame that rotates coincident with the circular motion of the primaries, a system of differential equations that describes the motion of the third body is written as

$$\ddot{x} = \frac{\partial U^*}{\partial x} + 2\dot{y}, \quad \ddot{y} = \frac{\partial U^*}{\partial y} - 2\dot{x}, \quad \ddot{z} = \frac{\partial U^*}{\partial z}, \tag{1}$$

*The acronyms CR3BP and CRP as well as the forms "restricted problem" and "circular restricted three-body problem" are all adopted and used interchangeably throughout.

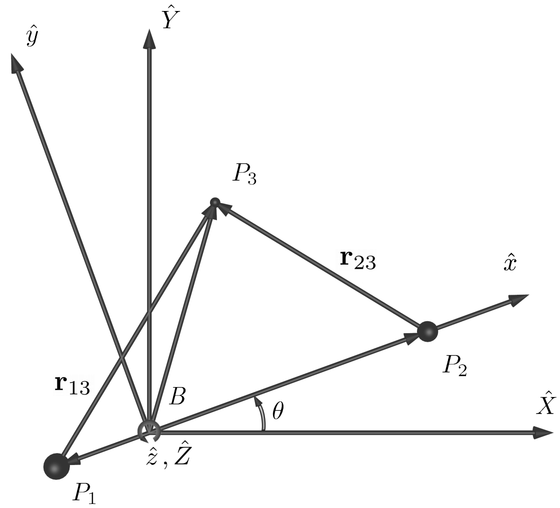


Figure 1: A pictorial schematic of the circular restricted three-body problem

with the pseudo-potential function defined as

$$U^* = \frac{1-\mu_{\text{CRP}}}{r_{13}} + \frac{\mu_{\text{CRP}}}{r_{23}} + \frac{1}{2}(x^2+y^2). \tag{2}$$

First derivatives in x and y appear in the equations of motion as a result of the Coriolis acceleration, and typical formulations incorporate nondimensionalization by characteristic system quantities in length, time and mass.

The equations of motion in the restricted problem, consistent with Szebehely,[14] admit a single constant of the motion in the rotating frame. This *Jacobi integral* is defined as

$$C_J = 2U^* - v^2, \tag{3}$$

where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, that is, the square of the magnitude of the velocity. The Jacobi constant enables a reduction of order in the problem, and frequently plays an important role in the definition of maps. The integral also reveals boundaries on the motion of the third body in the restricted problem; such insight is useful for validating the accuracy of numerical simulations and developing heuristics for trajectory and maneuver design.

The restricted problem represents a model of sufficient complexity to exhibit regions of both chaotic and relatively ordered behavior. Fundamental solutions such as libration points, periodic orbits and invariant manifolds emerge from the elegant mathematical approaches developed, in part, to study this problem. Generally, the focus of an analysis in this model is understanding and exploiting the underlying dynamics to identify useful trajectory arcs. The CR3BP is frequently used to yield first-order trajectory design solutions, and the features revealed through analysis of it as a dynamical system supply substantial insight into the associated design space.

IMPLEMENTATION DETAILS

Astrogator's numerical propagation model follows a modular execution strategy based upon the concepts of a "propagator" and a "numerical integrator". While these concepts or terms are part of the general astrodynamics vernacular, they are used specifically within Astrogator. Both objects follow the STK component paradigm and exist independently of a specific spacecraft or orbit model. Consequently, these components can be customized and configured as needed to support a broad variety of possible applications. Within the context of the CR3BP, a special propagator component has been introduced that appropriately constrains user choices. For example, the selectable gravity model offers only the associated circular restricted model propagator function—no two-body, gravity field or third-body perturbations allowed! Other propagator functions are allowed such as solar radiation pressure, general relativity and others. Propagation of the state transition matrix is also determined by the addition of its associated propagator function. Certain combinations are given special handling to best numerically condition the propagation when additional choices are added to the propagator. Likewise, the numerical integrator component embedded within the propagator has also been customized by appropriately limiting options that are not relevant to CR3BP analysis.

Prior to evaluation and simulation, internal mechanisms associated with the mission control sequence identify if the CRP functions are to be evaluated. If so, the Astrogator state is prepared to account for this evaluation. The Astrogator internal state holds various data essential to propagating and recreating the spacecraft ephemeris. Some of this data are context dependent, such as maneuver related properties. Consequently, such data are only updated within the particular context where they are relevant. This context-dependent state element update is the case for propagation associated with the CR3BP. If the CRP function is to be evaluated, the Astrogator initial state (however it has been defined by the user, then converted and stored internally in its STK Central Body Inertial, or CBI, representation) is transformed into its CRP representation. The CRP representation of the initial state is set into the Astrogator state vector in reserved space used only when propagating under restricted problem dynamics; the CR3BP mass parameter is also set into the state vector at this time. Finally, internal flags are set to indicate that a circular restricted force model will be used and whether the State Transition Matrix (STM) must be evaluated.

As execution of the MCS begins, each propagator function is processed in turn, building up the acceleration acting on the spacecraft. With each major step of the numerical integrator, many function evaluations are performed. It is during this chain of function evaluations that the newly implemented CRP equations of motion may be invoked and, depending on selected options, the state transition matrix functions may also be executed in response to the CRP evaluations. Evaluation of the CR3BP equations of motion represents a seemingly straightforward process, and this is typically the case when the simulation environment has limited preconditions. However, the "system" presented by the Systems Tool Kit architecture imposes some such preconditions that require careful handling. The strategy employed has been one of adapting the system *as well as* adapting the evaluation in a way that favors the most consistent numerical results, both in terms of self-consistency as well as in terms of consistency with external simulation, at the end of the process.

Astrogator CR3BP equations of motion implementation

The implementation of the CR3BP equations of motion relies on instantaneous transformations to produce a hybrid system that evolves under these equations of motion, but redefines the system with each time step, accounting for all aspects of the motion of the secondary. If the secondary body in the three-body system follows natural motion (i.e., non-circular), the associated motions will be incorporated into the instantaneous transformations inducing librations and pulsations in the rotating frame inconsistent with the CR3BP dynamics. However, if the secondary motion for the system is appropriately defined (i.e., moving on a circular orbit about the primary, etc.) the result is

Table 1: Transformation Matrices Employed in CR3BP Model Evaluation

Matrix	Dimension	Description
${}^{\mathrm{I}}\mathbf{R}^{\mathrm{R}}(t)$	6×6 or 9×9	Rotation matrix from the CR3BP rotating frame to the STK CBI frame
${}^{\mathrm{I}}\mathbf{S}^{\mathrm{R}}(t)$	6×6 or 9×9	A diagonal scaling matrix with the position, velocity and acceleration (in some cases) scale factors comprising the diagonal
${}^{\mathrm{I}}\mathbf{H}^{\mathrm{R}}(t)$	6×1 or 9×1	A vector to shift the state from the barycenter to the primary center
${}^{\mathrm{R}}\mathbf{R}^{\mathrm{I}}(t)$	6×6 or 9×9	Rotation matrix from the STK CBI frame to the CR3BP rotating frame
${}^{\mathrm{R}}\mathbf{S}^{\mathrm{I}}(t)$	6×6 or 9×9	A diagonal scaling matrix with the position, velocity and acceleration (in some cases) reciprocal scale factors comprising the diagonal
${}^{\mathrm{R}}\mathbf{H}^{\mathrm{I}}(t)$	6×1 or 9×1	A vector to shift the state from the primary center to the barycenter

consistent with the circular restricted model. Thus the implementation of the CR3BP in Astrogator assumes a guided framework where the environment is properly and precisely configured. Aside from these considerations (i.e., the noted adaptations of the STK system discussed in the software system preparation section), the numerical implementation is generally consistent with typical numerical integration approaches. The process is principally one of frame transformations performed in a specific order to prepare the position and velocity state variables to be utilized as inputs for the dynamical model. Once appropriately cast, the model is evaluated and the necessary quantities retained. Subsequent transformations are performed based on the propagator definition.

The CR3BP system is defined based upon the configuration of the propagator function. The central body for the propagator serves as the system primary, P_1, and the user must specify which STK central body is to act as the restricted problem secondary, P_2. The STK system has been configured to allow the choice of secondary bodies to include child or parent bodies of the primary. Given a CR3BP system, the transformation matrices are summarized in Table 1. In the table, the superscripts $^{\mathrm{I}}$ and $^{\mathrm{R}}$ imply notional relevance in dealing with two fundamental systems: STK's internal, dimensional Central Body Inertial (CBI) system and the CR3BP nondimensional, rotating, barycentric system. The superscripts not only represent the associated reference systems but they also generally convey the directional behavior of the transformation, based on context, read from right to left as a matrix (or vector) is premultiplied (or summed) against a state vector. The rotation matrices are formed consistent with STK's Rotating Libration Point (RLP) systems,[15] and incorporating similar notions to those found in Anderson.[16] They are used to transform position and velocity state variables in their 6×6 forms or position, velocity and acceleration in their 9×9 forms. In each case, all vector derivatives and kinematic considerations are incorporated directly into the rotation matrices. The scaling matrices are comprised of the position variable scaling factor on the diagonal of the upper-left 3×3 submatrix and the velocity variable scaling factor on the diagonal of the lower-right 3×3 submatrix in the case of only position and velocity transformation. In the case where acceleration is also transformed, the velocity scale factor comprises the central 3×3 diagonal submatrix of the larger 9×9 matrix, and the acceleration variable scale factor populates the diagonal of the 3×3 lower-right diagonal submatrix. The direction of scaling (up or down) determines whether the scaling factors in the matrix are either simply the characteristic quantities for the system or their reciprocals. Finally, the shift vectors are simply added to the state vector to translate the base point for the position components from the primary center to the system barycenter or vice versa, as appropriate.

Beginning with the position and velocity state components in dimensional form ($_{\mathrm{D}}$) and expressed in STK's CBI ($^{\mathrm{I}}$) coordinate system, designated $\mathbf{X}_{\mathrm{D}}^{\mathrm{I}}(t)$, as input to the model, the following sequence of operations is performed: rotate, scale, shift. These steps are performed in the following equations by first rotating the state in Equation 4, then nondimensionalizing in Equation 5.

$$\mathbf{X}_{\mathrm{D}}^{\mathrm{R}}(t) = {}^{\mathrm{R}}\mathbf{R}^{\mathrm{I}}(t)\mathbf{X}_{\mathrm{D}}^{\mathrm{I}}(t) \tag{4}$$

$$\mathbf{X}_{\mathrm{ND}}^{\mathrm{R}}(t) = {}^{\mathrm{R}}\mathbf{S}^{\mathrm{I}}(t)\mathbf{X}_{\mathrm{D}}^{\mathrm{R}}(t) \tag{5}$$

$$\mathbf{x}_{\mathrm{ND}}^{\mathrm{R}}(t) = {}^{\mathrm{R}}\mathbf{H}^{\mathrm{I}}(t) + \mathbf{X}_{\mathrm{ND}}^{\mathrm{R}}(t) \tag{6}$$

In Equation 6, the final step of shifting the state to the barycenter results in a traditional barycentric, nondimensional ($_{\mathrm{ND}}$), rotating ($^{\mathrm{R}}$) CR3BP state, where the lower case vector notation, i.e., $\mathbf{x}$, has been adopted for this overall state form.

Given a state posed in the native system of the CR3BP, the force model is simply evaluated* consistent with Equation 1. This evaluation yields the CR3BP acceleration, $\mathbf{a}_{\mathrm{ND}}^{\mathrm{R}}(t)$. With the acceleration computed, a 9×1 vector is constructed, $\tilde{\mathbf{x}}_{\mathrm{ND}}^{\mathrm{R}}(t)$, comprised of position, velocity and acceleration components. The restricted problem position and velocity state variables are also committed to the Astrogator state for potential use later in computing various quantities upon demand (e.g., C_J). To this point, the transformation matrices and vectors have been either 6×6 or 6×1. The addition of acceleration to the state vector invokes the necessity of the 9-dimensional counterparts. With the 9-dimensional state vector, the inverse transformation process to that performed in Equations 4–6 is executed: shift, scale, rotate.

$$\tilde{\mathbf{X}}_{\mathrm{ND}}^{\mathrm{R}}(t) = {}^{\mathrm{I}}\mathbf{H}^{\mathrm{R}}(t) + \tilde{\mathbf{x}}_{\mathrm{ND}}^{\mathrm{R}}(t) \tag{7}$$

$$\tilde{\mathbf{X}}_{\mathrm{D}}^{\mathrm{R}}(t) = {}^{\mathrm{I}}\mathbf{S}^{\mathrm{R}}(t)\tilde{\mathbf{X}}_{\mathrm{ND}}^{\mathrm{R}}(t) \tag{8}$$

$$\tilde{\mathbf{X}}_{\mathrm{D}}^{\mathrm{I}}(t) = {}^{\mathrm{I}}\mathbf{R}^{\mathrm{R}}(t)\tilde{\mathbf{X}}_{\mathrm{D}}^{\mathrm{R}}(t) \tag{9}$$

The resulting 9-dimensional state vector, $\tilde{\mathbf{X}}_{\mathrm{D}}^{\mathrm{I}}(t)$, is the CBI position, velocity and acceleration, with only the acceleration components representing "new information". At this point, the CBI position and velocity allow for consistency checks on the initial inputs to the model, while the CBI acceleration components are the basic necessary outputs for integration to proceed within STK's general numerical integration framework.

First-order partial derivative evaluations

In the case where the state transition matrix is to be computed, the first-order partial derivatives must be evaluated. Whether the STM is to be computed or not is determined by user inclusion of the STM propagator function on the numerical propagator component. Additional logic is imposed to determine in which frame to integrate the variational equations. If the propagator contains only the CR3BP and STM propagator functions, the partial derivatives are maintained in their native restricted problem representation to minimize numerical artifacts introduced through additional transformations. Otherwise, if additional forces are considered through the inclusion of propagator

*Several alternative formulations were considered including implementing the CR3BP equations of motion dimensionally or in a primary-centered form. Each alternative option yielded numerical results less consistent against various validation efforts and self-consistency checks than evaluation in the native CR3BP formulation posed here.

functions like, for example, solar radiation pressure*, the partial derivatives must be transformed into the common working frame for those other forces, CBI, before the variational equations are integrated. The transformations of the partial derivatives are noted through observation to introduce additional numerical bias upon integration. This numerical discrepancy is such that partials that have been transformed to CBI and integrated produce an STM when transformed back to the rotating frame that disagrees, over time, with an STM produced from natively integrated partial derivatives. While generally small, consistent with numerical integration error buildup, the disagreement does grow with simulation time and is therefore avoided when not including additional forces. The inclusion of additional forces, however, imposes the transformation as other options are prohibitive.

The simplest case for the evaluation of the first-order partial derivatives is when they are retained in their native rotating form for integration. However, since the remainder of the variables in the Astrogator state vector are typically integrated in dimensional time consistent with the process described in the previous section, the partials do require dimensionalization, even in this case. Consider the following matrix of (nondimensional) first-order partial derivatives:

$$\mathbf{A}_{\mathrm{ND}}^{\mathrm{R}}(t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}_{\mathrm{ND}}^{\mathrm{R}}} = \begin{bmatrix} \frac{\partial \mathbf{v}}{\partial \mathbf{r}} & \frac{\partial \mathbf{v}}{\partial \mathbf{v}} \\ \frac{\partial \mathbf{a}}{\partial \mathbf{r}} & \frac{\partial \mathbf{a}}{\partial \mathbf{v}} \end{bmatrix}, \tag{10}$$

where $\mathbf{F}$ represents the first-order system of differential equations corresponding to Equation 1, $\mathbf{x}_{\mathrm{ND}}^{\mathrm{R}}$ is the position-velocity state vector consistent with the left-hand-side of Equation 6 and $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are the position, velocity and acceleration 3-vectors, respectively. Dimensionalization of the partial derivatives in this matrix in preparation for integration is accomplished by dividing each of the elements of the four submatrices by the characteristic time parameter, t^*, of the CRP system, its square or unity, as appropriate (in some cases, an operation is naturally unnecessary and, therefore, not performed).

$\frac{\partial \mathbf{v}}{\partial \mathbf{r}}$ elements: divide by t^*

$\frac{\partial \mathbf{v}}{\partial \mathbf{v}}$ elements: divide by 1

$\frac{\partial \mathbf{a}}{\partial \mathbf{r}}$ elements: divide by t^{*2}

$\frac{\partial \mathbf{a}}{\partial \mathbf{v}}$ elements: divide by t^*

If additional forces are to be considered while integrating the STM, then the partial derivatives must be converted to the STK CBI system to match the partial derivative computations for the other forces. In this case, the dimensionalized matrix of partial derivatives is transformed as

$$\mathbf{A}_{\mathrm{D}}^{\mathrm{I}}(t) = {}^{\mathrm{I}}\mathbf{R}^{\mathrm{R}}(t)\mathbf{A}_{\mathrm{D}}^{\mathrm{R}}(t){}^{\mathrm{R}}\mathbf{R}^{\mathrm{I}}(t), \tag{11}$$

where both rotation matrices are the 6×6 position-velocity rotation matrices. The partial derivatives with respect to position are carried through for incorporation with any other force model partial derivatives to be integrated and produce the resulting STM.

Upon completion of the numerical propagation of the CRP, the Astrogator state vector contains its typical data, including CBI position and velocity components and any other contextual data. In this particular case, CRP state variables are also populated for each step in the ephemeris as well as the CRP mass parameter. Further, if the STM was propagated it was produced either in the native

*Inclusion of additional forces into the model is an allowance afforded by Astrogator's modular paradigm. However, if the user chooses to do so, the resulting system will, of course, increasingly depart from the fundamental mathematical character consistent with the CR3BP.

restricted problem system or in CBI. If computed in the native CRP system, the STM is transformed for storage in the state vector into CBI for consistency—at the present time, the STM is always saved into the Astrogator state vector in CBI.

SOFTWARE SYSTEM PREPARATION

To incorporate the lower-fidelity dynamical model represented by the CR3BP into a high-fidelity tool like STK, appropriate adjustments are required. While the fundamental mathematics associated with the model have been implemented directly into STK for use within Astrogator, those mathematics rely on the same assumptions in STK as they would elsewhere. Namely, that the three-body system is consistent with the CR3BP definitions. In this case, that means the system must also be adjusted to match the mathematical model. Primarily, these adjustments involve adapting the environment to be consistent with the definitions given by Szebehely[14] and reflecting common modifications consistent with Howell[17] and others. The process detailed in this section has been captured to varying degrees in other AGI media including a tutorial[18] as well as a high-level video.[19] The inclusion of the process here provides a mechanism for additional elaboration and clarification. At the present time the steps outlined in the following two subsections must be carried out by the user. A point of future effort is to remove this burden from the user and provide a tool within the STK system to perform these operations. Regardless, the details are illustrative and relevant to any future automated process.

Creating a consistent STK system

The Systems Tool Kit represents not only a suite of software capabilities to interact with time-dynamic systems but also a system itself. It is comprised of multiple subsystems that work in concert to simulate various physical phenomena. One critical component of the system is that of the central body. While an STK scenario typically holds a default central body, which can be selected upon scenario creation, this default does not restrict the environment to analysis only on and about that body. Rather, the default central body most often serves simply as the default for new object creation and other settings. Astrogator is a particular subsystem within STK that often simultaneously takes advantage of multiple central bodies, each of which may be configured independently by the user. The central body components in the STK Component Browser serve as user level interfaces to lower-level STK central body constructs, and provide a mechanism for a user to customize or create a new central body. It is this capability that is exploited to appropriately configure the CRP system.

A fundamental assumption of the circular restricted three-body problem is that the two massive primaries mutually orbit on circular orbits about their common barycenter. As no real-world systems satisfy this assumption under the JPL ephemerides used to model the orbital motions of the STK bodies, a suitable alternative means to satisfy the assumption must be made if the CR3BP model is to be used. The assumption can be satisfied by a secondary in a circular orbit about the primary in the inertial system. Using this as the basis for bridging the gap between STK's environment and the CRP model, the key step is to create the appropriately defined secondary central body. Then, the CR3BP propagator can use the physical and orbital parameters of the secondary along with those of the primary, to fully characterize the CR3BP system. The process begins with creating the central body to act as the secondary. The user:

1. Duplicates an existing central body in the STK Component Browser (ideally one with minimally predefined data like the "asteroid template").

2. Enters physical properties of the secondary, specifying its gravitational parameter and parent.

3. Optionally defines the attitude of the secondary (this allows for visualization in a secondary "fixed" frame).

4. Defines the secondary's orbit to be circular, consistent with desired parameters for the CRP system. All orbital element rates are zero except that of the anomaly parameter, which defines the rotating frame rate.

5. Configures STK to show the new central body graphically.

The pattern of duplicating an existing component in the Component Browser to create a new component has been long established within STK and is generally effective; however, this approach can leave multiple settings "to be unset" in the case of a central body. Thus it is advised to use the recently created "asteroid template". Upon creation of the new body, setting its two-body gravitational parameter (GM_{2b}) is critical as this will be used in conjunction with the primary's gravitational parameter to define the CR3BP mass parameter, μ_{CRP}, as

$$\mu_{\mathrm{CRP}} = \frac{\mathrm{GM}_{\text{2b, secondary}}}{\mathrm{GM}_{\text{2b, primary}} + \mathrm{GM}_{\text{2b, secondary}}}. \tag{12}$$

Further, the specification of the secondary's parent signals to the CR3BP propagator and associated propagator function that it (the secondary) may be selected in combination with the central body for the propagator, which acts as the primary. Only children or the parent of the primary may be selected when choosing the secondary for the CR3BP propagator function. Specifying the attitude of the newly defined secondary enables advanced visualization and analysis, but is not required for the CR3BP force model to work. The specification of the analytic orbital parameters for the secondary determines the initial system configuration and epoch. This specification allows for significant flexibility in defining the CR3BP system to match a natural system at some epoch. Finally, the configuration of various graphical settings allows for visualization of the secondary's orbit as well as, perhaps, adding a 3D model onto the central body.

Creating a working coordinate system

The next major step in configuring the STK system involves invoking another STK subsystem, the Analysis Workbench (AWB). The goal at this step is to configure a convenient working coordinate system, and involves using the definitions of the CR3BP to define geometric constructs to build the coordinate system. The process involves creating:

1. A displacement vector between the primary and secondary using actual positions (i.e., not using STK's adjustments for light time delay).

2. An aligned and constrained unit-vector axes set with the primary-secondary vector direction serving as the "x" direction, and the secondary's orbit angular momentum serving as the "z" direction; the "y" direction follows automatically to complete the dextral, orthonormal triad.

3. A primary-centered coordinate system using the unit-vector axes from the previous step with a reference point at the primary center.

4. A point for the barycenter along the "x" axis at the distance of $\mu_{\text{CRP}} \cdot l^*$ from the primary center, with l^* being the characteristic distance of the CRP system.

5. A barycentered coordinate system using the same unit-vector set with a reference point at the system barycenter.

Each of these geometric constructs is useful for particular aspects of analysis in the CR3BP, but the principal element is the barycentered coordinate system. With the coordinate system defined, the user may now specify initial conditions in Astrogator consistent with the CR3BP system (in dimensional units). Further, Astrogator stopping conditions and calculation objects can make use of the new coordinate system.

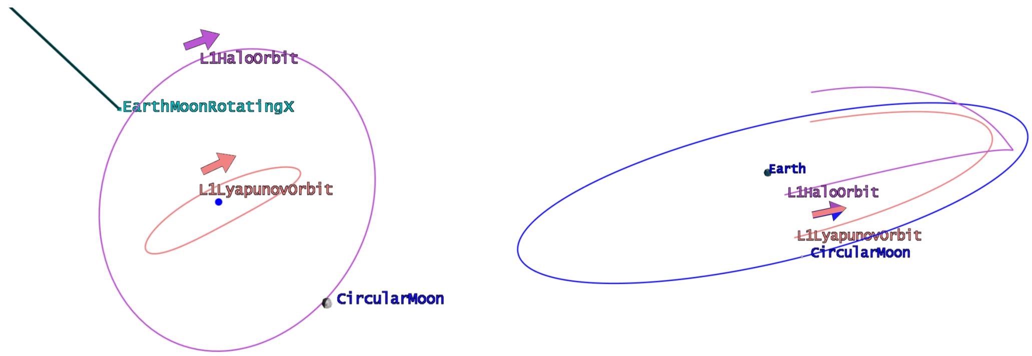
With the STK system configured appropriately and a working coordinate system set up, the CR3BP force model and attendant propagator can be used to effectively simulate solutions under the restricted problem dynamics. An Astrogator satellite object is configured with a propagator component using the force model against the primary-secondary system. Various Astrogator calculation objects can be configured to report on data in the new coordinate system. The Jacobi constant is also available for computations against ephemeris propagated with the CR3BP propagator. Finally, the specific Astrogator satellite's graphics properties can be configured such that the orbit system for display is the CRP rotating frame. These capabilities represent the basic functionality to simulate trajectories in the restricted problem. Additional capabilities remain under development.

VALIDATION

To ensure validity in the internal implementation of the model within the unique STK Astrogator simulation environment, various accuracy checks and strategies are employed. Comparisons with literature as well as self-consistency of results represent two approaches. Evaluation in external simulation environments and across numerical integration schemes are additional choices. Aspects of these approaches as well as others are invoked to verify and validate the CR3BP model in STK, and particular details are offered.

A common and simple check graciously afforded by the CR3BP is a check of the time history of the integral of the motion, the Jacobi constant, C_J. Although a fairly straightforward quantity to compute, the Jacobi constant is often nontrivial to recover from operational software as it requires the state variables to be posed in nondimensional, rotating coordinates consistent with the CR3BP model. With the addition of the model natively into the Astrogator framework, the Jacobi constant is now also available for use in the tool with this specific model. In Fig. 2, two periodic orbits about L_1 in the Earth–Moon CR3BP are depicted, a planar Lyapunov orbit (period: 12.46 days) in peach and a three-dimensional halo orbit (period: 12.28 days) in lavender. In the figure, the direction of motion is generally indicated by arrows and two views are included: a rotating frame view and an inertial view. In Fig. 3, the absolute difference from the initial Jacobi constant (ΔC_J), a function of numerical simulation error, is plotted along the single revolutions of each orbit depicted in Fig. 2. Although the difference is non-zero it remains small, an expected behavior of a constant derived from numerically integrated state data. These results reflect two data points associated with orbits that are relatively well behaved numerically, and serve as an introduction to a more general process.

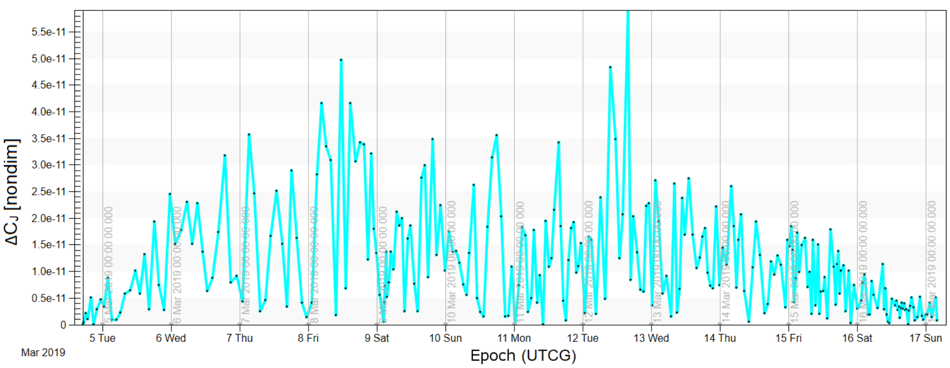
The preceding process is now extended across two families of orbits and specific details associated with the process in terms of producing the families as well as numerical integration schemes are provided. Aggregate results and the relative efficacy of those results are also discussed. The



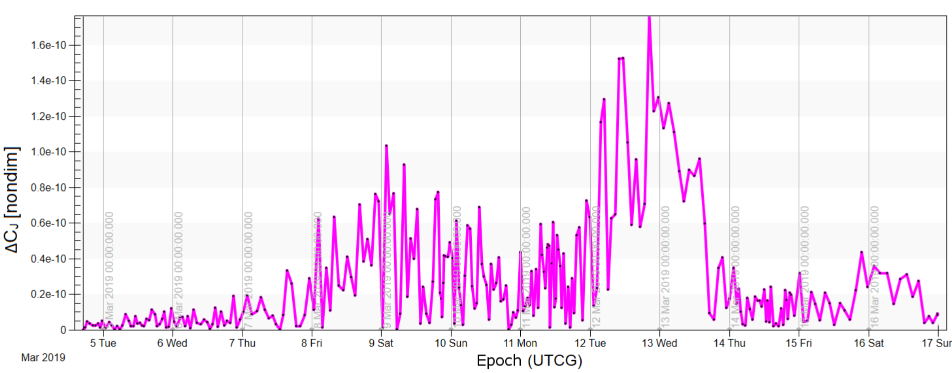
(a) Rotating view

(b) Inertial view

Figure 2: Earth–Moon Lyapunov and halo orbits about L_1 constructed natively in Astrogator



(a) Lyapunov orbit



(b) Halo orbit

Figure 3: Jacobi constant variation associated with numerical integration of the orbits from Fig. 2

analysis is expanded and applied to the L_1 Lyapunov and southern halo families under the following methodology. System parameters are taken from the three-body system constructed in STK and used to establish an independent simulation. This simulation is carried out in MATLAB with a standard implementation of the first order differential equations of motion from Equation 1. In MATLAB the numerical integration scheme `ode113` is used with integration tolerances (absolute and relative) of 1×10^{-12}. The initial Lyapunov orbit is constructed from linear analysis about the L_1

point and the family is continued via Pseudo-Arclength Continuation (PALC)[20,21] with individual orbits being corrected with multiple shooting on four control points. Inspection of the eigenvalue structure of the monodromy matrix for constituent family orbits reveals bifurcations with other families. The bifurcation leading to the L_1 halo families is followed along the southern branch, again via PALC. These orbits are depicted in Figs. 4–5, where in Fig. 4a the Lyapunov orbits are depicted in a color scale ranging from cerulean through shades of blue and pink to fuchsia. This coloring is normalized among the included representative orbits and corresponds to closest approach distance to P_2. In Fig. 4b, a zoomed-in region near P_2, denoted "CircularMoon", is included along with an x–y perspective projection of the southern halos superimposed on the Lyapunov orbits to convey a relative sense of scale. The Earth, P_1, is also visible in some of the images. Figure 5 offers several perspective projections of the L_1 southern halo family, constructed within STK. Simulation of the family is terminated in the range of orbits denoted Near Rectilinear Halo Orbits (NRHO)[22,23] with the member produced via continuation whose closest approach of P_2 does not impact the surface (altitude: ≈86 km). Again, the representative halo orbits are colored by closest approach to P_2. General orbital motion in the rotating frame is consistent in the orbits of these families as with the orbits included in Fig. 2a. The generally central cyan point (unlabeled in several views to reduce visual clutter) is L_1.

Subsequent to the family continuation and corrections process within MATLAB, the orbits are repropagated from initial states, terminating on their previously determined periods. The Jacobi constant is calculated and maximum variations are stored from the numerical integration within this framework. The computation of the maximum variation of the Jacobi constant along a given orbit is computed as:

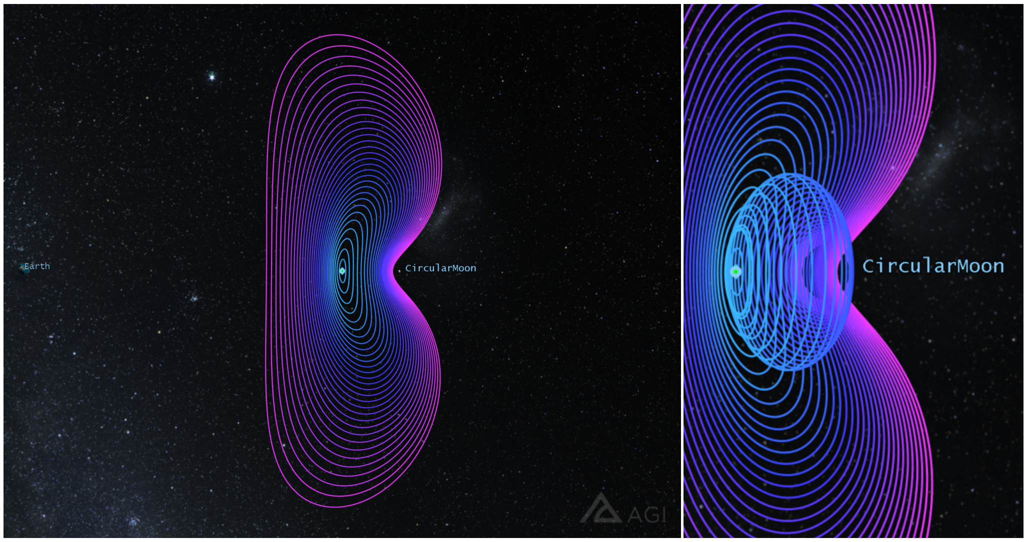
$$\text{Max}\left(\Delta C_J\right) = \max\left(\text{abs}\left(C_{Ji} - C_{J\text{o}}\right)\right)\ \forall i, \tag{13}$$

where $_i$ indicates an indexed state within the orbit's ephemeris with $_\text{o}$ corresponding to the first state. Further, consistent with the repropagated ephemeris, a measure of orbit closure is calculated as:

$$|\Delta \mathbf{X}| = |\mathbf{x}_{\text{tf}} - \mathbf{x}_{\text{to}}|_2 \tag{14}$$

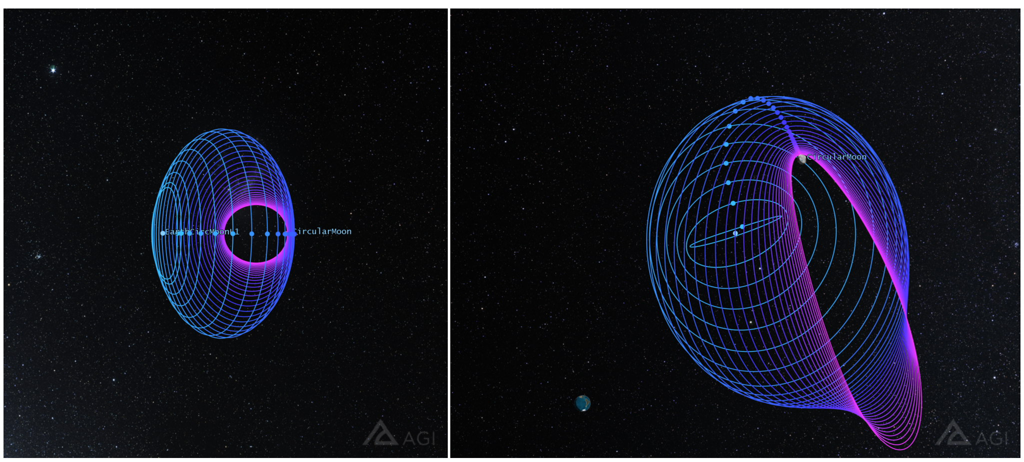
That is, the closure measure is simply the l^2-norm of the vector difference between the initial state and the final state of the orbit.

Given the converged orbits, their initial conditions, periods, Jacobi constants and closure measures, the analysis shifts back into Astrogator. The STK Astrogator Component Object Model (COM) Application Programming Interface (API) is used through MATLAB to attach to the STK scenario previously constructed for use with the restricted problem. In this case, the scenario uses the Earth as P_1 and a user-configured "Moon" placed on a circular orbit denoted "CircularMoon" in Figs. 4–5. The analytical orbital parameters for the lunar orbit are consistent with those given in the system configuration tutorial.[18] The COM API is further utilized to add Astrogator satellites to the scenario, populate those satellites with initial state and propagate segments, and configure the segments to hold the initial conditions and other parameters from the MATLAB orbit family simulations. Calculation objects are added to the propagate segments and the orbit system graphics settings are configured for each satellite to depict the orbits in the rotating frame. Finally, the mission control sequence for each satellite is executed and the calculation object data providers are evaluated to compute the Jacobi constant along the orbits, its maximum variation and the orbit closure measure. The visuals comprising Figs. 4–5 result from Astrogator propagations. Although available from Astrogator's target sequence capabilities, no numerical corrections are performed; the goal is to simply propagate the orbits and evaluate the associated efficacy of the propagation.

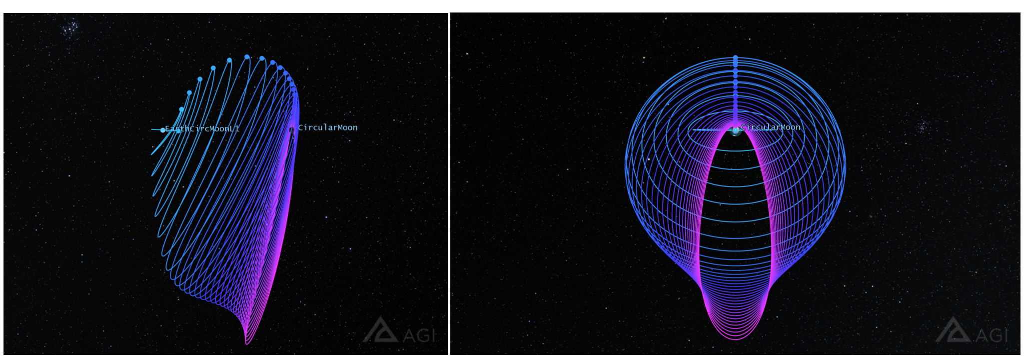


(a) L_1 Lyapunov orbit family (b) L_1 halo family for comparison

Figure 4: L_1 Lyapunov and halo orbit families projected onto the x–y plane



(a) Rotating x–y perspective projection (b) 3D perspective view

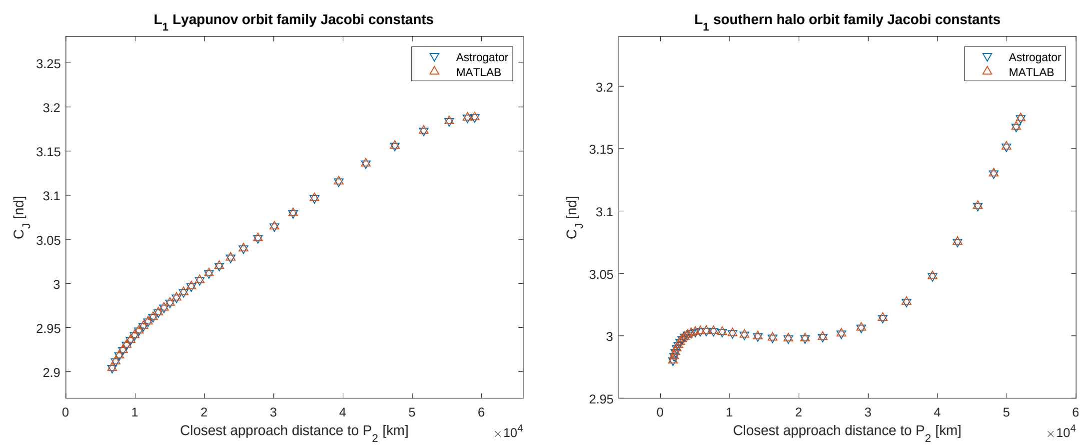


(c) Rotating x–z perspective projection (d) Rotating y–z perspective projection

Figure 5: Southern L_1 halo orbit family

Propagation in Astrogator is accomplished with minor modifications to the default integration settings. The selected numerical integration scheme is the default `RKF7th8th` integrator, an adaptive step size Runge–Kutta–Fehlberg 7th order scheme with 8th order error control. The requested error tolerances are decreased to 1×10^{-13} and 1×10^{-15} (absolute and relative, respectively). These error controls are enforced on the **dimensional** state within STK, a consideration that becomes immediately apparent in the resulting analysis. In this case, the user-customized three-body propagator includes the state transition matrix propagator function because the MATLAB simulation includes the STM, however, no STM analysis is included for this effort as relevant data have not yet been exposed for Astrogator. Under the different numerical integration parameters, it is expected that results will generally disagree numerically between the two simulations. Nevertheless, the aggregate behavior is consistent as evidenced immediately by the reproduction of the orbit families under simple propagation. Figures 4–5 represent the initial validation of the STK Astrogator model—it reproduces the solutions from the external simulation.

Subsequent comparison between the two simulations helps to establish how well the model captures the dynamics. Direct comparison between the Jacobi constant of representative orbits from both families is offered in Fig. 6. In this and subsequent figures, the horizontal axis is marked in terms of distance from P_2, thus the orbits closest to L_1 in each family are found on the right-hand side of the plots. The Jacobi constant is visually consistent for the orbits from both simulations.



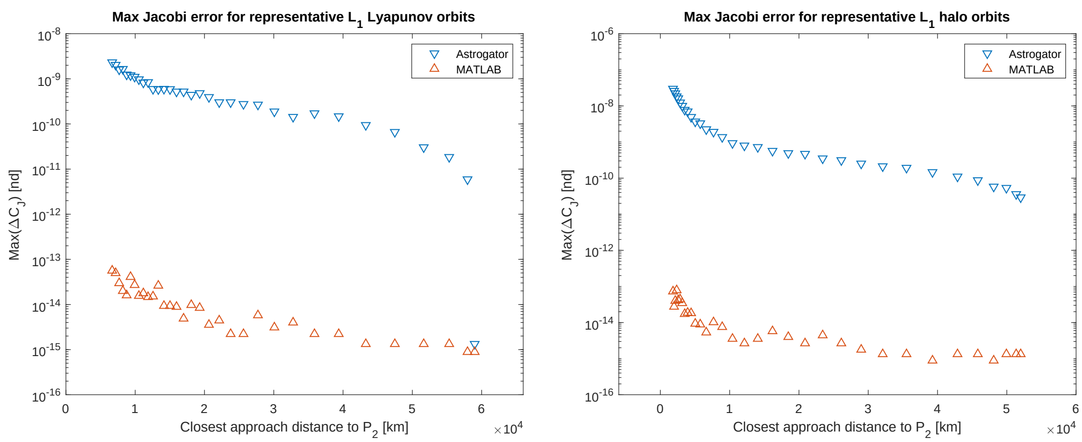
(a) (b)

Figure 6: Jacobi constant values for the representative orbits along the two families in Figs. 4–5

Recall, the implementation of the circular restricted three-body problem within STK required a bridging of the gap between the simulation environment and the model. While this was accomplished with the goal of maximizing accuracy and consistency of the results, some tradeoff is observed. To utilize the existing STK and Astrogator paradigms such as STK's environmental analysis capabilities and Astrogator's calculation system for stopping conditions and data reporting, state transformations are necessary and these incur a numerical price. This is immediately apparent in Figs. 7–8, where the maximum Jacobi constant variation and the orbit closure measure for each orbit are plotted, respectively. In each case the MATLAB simulation achieves roughly five orders of magnitude greater accuracy. This is unsurprising since numerical integration error control is enforced on the dimensional state in STK; this error control is essentially five orders of magnitude looser than that imposed on the nondimensional state in the analogous MATLAB integration.

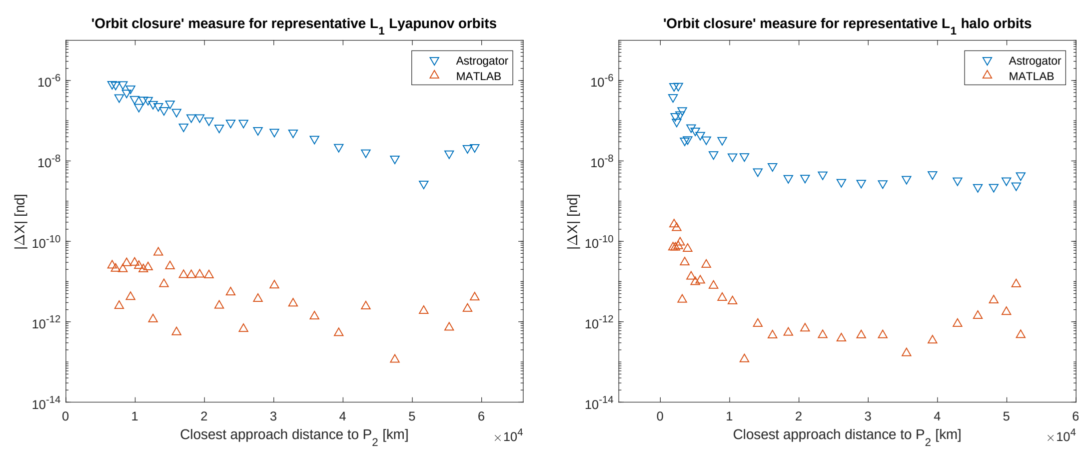
Two points of interest emerge from inspection of the Jacobi constant error results. The error grows dramatically in the context of the families in all cases with proximity to P_2. This is also true for individual orbits as their proximity to P_2 increases when inspected in plots similar to Fig. 3, in these cases the maximum error is concentrated near the close approach.

The generally higher degree of numerical error in the Jacobi constant for the Astrogator propagations may be understood in the context of the evaluation process. At each numerical integration step, the CBI state is transformed into a CRP state undergoing nondimensionalization. The CRP acceleration is evaluated and transformed into CBI (undergoing dimensionalization) for the integration step, at this time the CRP position and velocity are stored in the Astrogator state vector. Thus, the CRP state is held to the dimensional error control in the integration step. Depending on the magnitude of the characteristic quantities for the system, the associated numerical integration error in the CRP state will automatically inherit an equivalent bias.



(a) (b)

Figure 7: Maximum Jacobi constant variation from numerical integration of the orbits from Fig. 6



(a) (b)

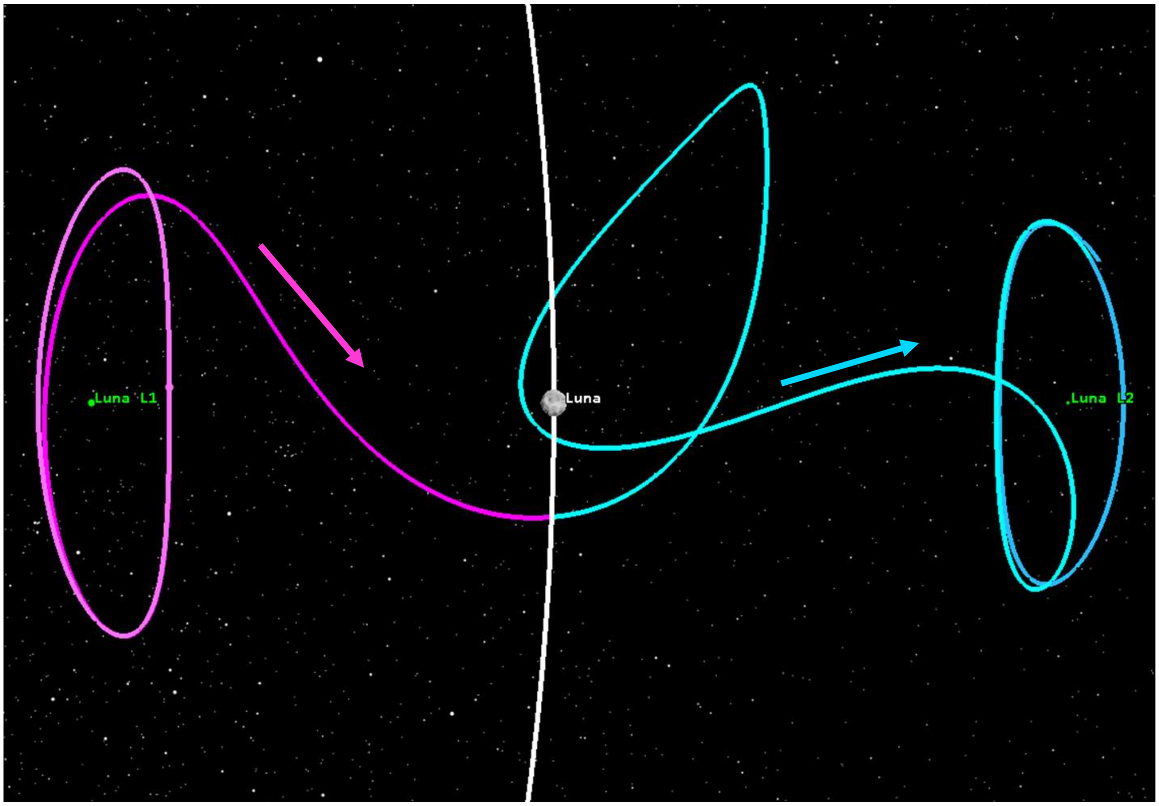
Figure 8: Closure measure after numerical integration of the orbits from Fig. 6

RAPID INGESTION, CONSTRUCTION AND CORRECTION OF SOLUTIONS

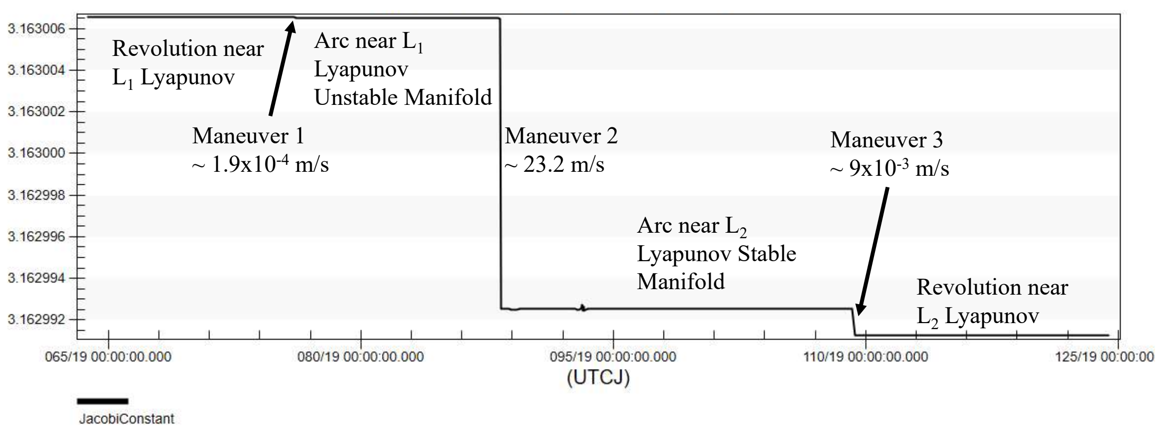
The use case in this section is constructed in an STK system similar to that produced for the validation work of the previous section. In this particular case the idealized "Moon" for the Earth–Moon CR3BP is denoted "Luna" as in the system configuration tutorial.[18] Given known parameters for an L_1 Lyapunov orbit ($C_J \approx 3.163007$) with the goal to insert into an L_2 Lyapunov orbit ($C_J \approx 3.162991$), initial guess arcs are seeded along the unstable manifold associated with the L_1 Lyapunov orbit and the stable manifold associated with the L_2 Lyapunov orbit. As the two orbits possess different energy levels, indicated by unequal Jacobi constant values, it is expected that one or more maneuvers will be required to transfer between the two orbits. The transfer solution is depicted in Figure 9 along with the time-history of the Jacobi constant.

In the transfer design, three maneuvers are placed in the itinerary to support the solution process and adjust the energy level. The first maneuver generally represents a perturbing step off the L_1 Lyapunov orbit onto the unstable manifold and facilitates numerical corrections; its magnitude is small: $\approx 1.9\times10^{-4}\frac{\text{m}}{\text{s}}$. Next, a match point between the unstable manifold arc from the L_1 Lyapunov orbit and the stable manifold arc from the L_2 orbit is targeted using the differential corrector at the $\Sigma : x = 1 - \mu_{\text{CRP}}$ hyperplane. At this match point the energy difference is accounted for by a second maneuver of $\approx 23.2\frac{\text{m}}{\text{s}}$. Finally, at the stable manifold interface with the L_2 Lyapunov orbit a third maneuver is performed to adjust the spacecraft state consistent with insertion into orbital motion associated with the L_2 Lyapunov orbit ($\approx 9\times10^{-3}\frac{\text{m}}{\text{s}}$). All three maneuvers are recalculated by targeting known states on the stable manifold and near the periodic orbit, and convergence is quickly met for tolerances set in postion at $\epsilon_{\text{pos}} < 1\text{km}$ and $\epsilon_{\text{vel}} < 1\frac{\text{m}}{\text{s}}$. This particular process illustrates the capability to transfer applicable knowledge of potential solutions into STK, rapidly cast the problem in Astrogator constructs and leverage the available numerical corrections schemes to form an end-to-end trajectory. While this example was produced with knowledge of existing orbits, all corrections occur within the STK Astrogator framework. Finally, after such a process, the resulting multi-body design is now available within Astrogator for additional analysis and advanced study in higher-fidelity dynamical models.

This particular use case is an example that highlights a mechanism for transitioning and recreating complete multi-body transfers in STK. Such capabilities, available in a tool like Astrogator, reinforce the applicability of the associated concepts. Traditionally, multi-body approaches have been studied heavily in academics, but their relevance continues to expand into industry. Access to associated capabilities in the tools used within the industry supports workforce training. This capability to analyze the CR3BP within STK enables training without requiring significant time and resources to create specialized scripts, and supports the application of these lessons in commonly-used platforms in new scenarios—thereby expanding access to knowledge from the multi-body community. Further, an example like this illustrates the process of seeding a complex design into the software with higher confidence as the level of effort to bring the toolset to the problem space is alleviated. In general, the expanded capabilities presented by recent development efforts and the present exercise to illustrate and capture these efforts in the literature both serve to clarify the overall process. These steps also help to discern and verify how to proceed when approaching multi-body trajectories with the CR3BP in the STK Astrogator framework.



(a) Transfer from L_1 Lyapunov to L_2 Lyapunov orbit



(b) Jacobi constant history for transfer

Figure 9: A multi-body transfer solution produced and corrected in STK Astrogator

CONCLUDING REMARKS AND FUTURE WORK

The ability to rapidly create and reproduce multi-body solutions within STK allows users to quickly bring their analyses to the point of considering other salient details of the system. While numerical tradeoffs exist associated with the incorporation of the CR3BP into the existing STK Astrogator architecture, these are often overshadowed by the natural goal of evolving solutions toward higher fidelity. These tradeoffs represent a potentially acceptable cost to enable typical extended STK Astrogator design and analysis capabilities. Additional options to mitigate the numerics by offering an implementation separated from the typical Astrogator mechanisms may be possible if a demonstrated need or desire is evidenced. More concretely, work is planned to offset much of the existing user incumbency to configure STK to work in the CR3BP model by offering tools to easily set up the STK environment for such operations.

With the addition of the CR3BP model natively into the STK Astrogator system, spacecraft trajectories can now be quickly, natively simulated under these dynamics. This option supports early trajectory design activities, and allows a pathway for users who spend their time and effort developing elegant and beautiful solutions to directly ingest such solutions directly into a procedure for model fidelity advancement. The Jacobi constant, directly available for CRP propagation, offers a quick validation against external simulation and provides a measure of internal consistency. Further, the accessibility of the Jacobi constant enables maneuver and trajectory design heuristics as in the case of the final example where the change in C_J at the manifold interface indicates a necessary impulse. Recognizing the need to adapt the STK system to the model enables a straightforward definition of the rotating coordinate system consistent with CR3BP definitions. While such a system configuration process was previously supported, it has been generally improved and the underlying model now exists to take advantage of it. All of these advancements are intended to increase the accessibility of multi-body trajectory design concepts, reduce the burden on typical users and provide additional capabilities in support of increasing mission concepts in cislunar space and other multi-body environments.

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