Light Time Delay and Apparent Position

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Introduction

The relative position of an object B with respect to an object A can be computed in a variety of ways, depending on the model of signal transmission between the two objects. When light time delay is not considered, the speed of light is considered infinite and there is no time difference between the transmission event and the reception event. Quantities that ignore light time delay are often termed 'true'; e.g., the true relative position of B with respect to A, computed as:

$$\mathbf{r}(t) = \mathbf{R}_B(t) - \mathbf{R}_A(t) \tag{1}$$

The term 'apparent' is used when the relative position vector accounts in some way for light time delay. The apparent position models signal transmission occurring at the finite speed of light so that a signal transmitted at time t is not received until $t+\Delta t$, where Δt is the light time delay (a positive number).

Light propagation models have a rich history, but for our purposes we only need be concerned with three kinematics models: (i) Galilean Relativity, (ii) Special Relativity; and (iii) General Relativity.

Galilean Relativity is by far the most widely known model, where space is completely separable from the concept of time. Space is modeled as a Euclidean space with the standard vector operations for a linear space; time is an absolute quantity known to all observers. Special Relativity models light propagation in such a manner that all inertial observers will measure the speed of light (in vacuum) as the same constant value c. Space is no longer separable from time; space-time is not a Euclidean space but instead a Minkowski space. Concepts that were once trivial now become more complicated: different inertial observers now disagree on simultaneity of events, on distances between objects, and even on how fast time evolves. However, light still propagates as a straight line in the spatial components. General Relativity goes one step further, removing the special status of inertial observers and introducing mass as generating the curvature of space-time itself. The light path deflects (curves) in the spatial components near massive objects.

Our goal in the modeling of light propagation is simply to include the first order corrections on Galilean Relativity caused by Special Relativity for signal transmission. Thus, we strive for accuracy to order β, where $\beta = v/c$, where v is the inertial velocity of a frame being considered. The light path then is a straight line in inertial space where the signal moves at constant speed c (i.e. gravitational deflection is ignored).

Computing Light Time Delay

We will first consider the light time delay for a signal transmitted from an object A to an object B. Later, we will consider the delay for a received signal at A.

Transmission from A to B

Consider an inertial frame F. Let $\mathbf{R_A}$ locate object A in F; let $\mathbf{R_B}$ locate object B in F. Let the relative position vector $\mathbf{r}$ be defined by

$$\mathbf{r}(t) = \mathbf{R_B}(t + \Delta t) - \mathbf{R_A}(t) \quad (2)$$

where t is the time of transmission from A and $t+\Delta t$ is the time of reception at B. The light time delay is Δt which will depend on t as well. Let $r = \|\mathbf{r}\|$ be the range between the objects. Then $\Delta t = r/c$.

One usually knows the locations for the objects A and B and computes the light time delay at time t through iteration. First, a value of Δt is guessed (often taken to be 0.0 or the last value computed at a previous time) and $\mathbf{r}(t)$ is computed. A new value for Δt is found from r/c and the procedure repeats. The iteration stops whenever the improvement in the estimate to Δt is less than the light time delay convergence tolerance. Typically, few iterations are required as the procedure converges very rapidly.

Reception at A from signal sent from B

In this case, the relative position vector is

$$\mathbf{r}(t) = \mathbf{R_B}(t - \Delta t) - \mathbf{R_A}(t) \quad (3)$$

where the signal is received by A at time t. The same procedure is used to find Δt, using $\mathbf{r}(t)$ above.

> NOTE: The light time delay Δt computed for the transmission from A and for reception at A are different, as is the relative position vector **r**.

The Inertial Frame

The choice of the inertial frame is important when computing light time delay, as it will affect the results. This is a consequence of Special Relativity. Let F and F' be two inertial frames with parallel axes. Let $\mathbf{v}$ be the velocity of F' with respect to F. In Special Relativity, time is not absolute but is instead associated with a frame: let t denote time in F and t' denote time in F'. For simplicity, assume the frames are coincident at $t=0$. Then the Lorentz transformation relating these two coordinate time values is

$$t' = \gamma\left(t - \boldsymbol{\beta}\bullet\frac{\mathbf{R}}{c}\right) \quad (4)$$

where **R** is the position vector of a location in *F* (measured from the origin of *F*), and

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c},\ \beta = \left\|\boldsymbol{\beta}\right\|,\ \delta = \sqrt{1-\beta^2}\ \text{, and } \gamma = \frac{1}{\delta} \quad (5)$$

The value *t′* is the value of time in *F′* for an event at time *t* at position **R** in *F*. Note that *t′* depends on both *t* and **R**.

Consider the case of transmission from object A, located at the origin of *F* at time *t=0*, to object B that receives the signal at time *t=Δt*. The value of *t′* at transmission is computed to be 0 (since both *t* and **R** are zero then). In *F*, the light time delay *Δt* is computed by solving

$$\left\|\mathbf{R}_{\mathbf{B}}(\Delta t)\right\| = c\,\Delta t \quad (6)$$

for *Δt*. Using the Lorentz transformation, the value of *t′* at reception is

$$t' = \gamma\left(\Delta t - \boldsymbol{\beta}\bullet\frac{\mathbf{R}_{\mathbf{B}}}{c}\right) = \gamma\,\Delta t\left(1-\hat{\mathbf{e}}_{\mathbf{R}_{\mathbf{B}}}\bullet\boldsymbol{\beta}\right),\ \text{where } \hat{\mathbf{e}}_{\mathbf{R}_{\mathbf{B}}} = \frac{\mathbf{R}_{\mathbf{B}}}{c\Delta t} \quad (7)$$

In *F′*, the light time delay is

$$\Delta t' = \gamma\,\Delta t\left(1-\hat{\mathbf{e}}_{\mathbf{R}_{\mathbf{B}}}\bullet\boldsymbol{\beta}\right) \quad (8)$$

To order *β*, *γ* is 1.0, so the difference *δt* in the computed light time delays between the two frames is

$$\delta t = \Delta t - \Delta t' = \Delta t\left(\hat{\mathbf{e}}_{\mathbf{R}_{\mathbf{B}}}\bullet\boldsymbol{\beta}\right) \quad (9)$$

The case of reception at A at time *t*=0 is analogous, producing the same result.

NOTE: The choice of inertial frame affects the computation of both the light time delay *Δt* and the apparent relative position vector **r**.

Choosing an Inertial Frame

Most space applications involve objects located near one central body. It is natural to associate a central body with an object. The natural inertial frame to use for modeling spacecraft motion near a central body is the inertial frame of the central body. (We use the term CBI for Central Body Inertial.) The CBI frame is a natural choice for the inertial frame for computing light time delay.

For objects that are far from their central body, however, the more appropriate inertial frame to model motion is a frame with origin at the solar system barycenter. This frame is used to model the motion of the central bodies themselves. This provides another choice for the inertial frame.

We take the view that CBI is the preferable frame for computing light time delay, but we want to insure that its use appropriately models the physics of the situation at hand. Thus, at the start of the light time delay computation we compute the difference δt between the use of the CBI and solar system barycenter frames. If this difference is less than the light time delay convergence tolerance, then either frame may be used to obtain the same level of accuracy—we choose the CBI frame because it is less expensive computationally. If the difference is more than the tolerance, we use the solar system barycenter frame knowing that it is a better model of an inertial frame in general.

Earth Operations

For the light time delay convergence tolerance of 50.e-5 seconds (i.e., 50 micro-seconds), objects located from near the Earth's surface to just outside the geosynchronous belt will use Earth's inertial frame for performing light time delay computations. Farther out than this, the solar system barycenter frame will be used. In particular, computations involving objects at the Earth-Moon distance will use the solar system barycenter frame for computation of light time delay.

Signal Path

With Δt and **r** determined from the light time delay computation performed in the inertial frame F, it is now possible to model the actual signal transmission (i.e., the path of the signal through F). The signal path is given by

$$\text{Transmit from A at } t: \quad \mathbf{s}(t+\tau) = \mathbf{R_A}(t) + c\tau\hat{\mathbf{e}}_\mathbf{r} \tag{10}$$

$$\text{Receive at A at } t: \quad \mathbf{s}(t+\tau-\Delta t) = \mathbf{R_A}(t) + c\left(\Delta t - \tau\right)\hat{\mathbf{e}}_\mathbf{r} \tag{11}$$

where $0 \le \tau \le \Delta t$ and $\tau=0$ locates the transmission event and $\tau=\Delta t$ locates the reception event. The apparent direction is given by

$$\hat{\mathbf{e}}_\mathbf{r} = \frac{\mathbf{r}}{r},\ r = \left\|\mathbf{r}\right\|,\ \mathbf{r} = \mathbf{R_B}(t+\sigma\Delta t) - \mathbf{R_A}(t) \tag{12}$$

where Δt is the light time delay computed in F and σ=1 in the case of transmission and σ=-1 for reception.

Aberration

Aberration is the change in the perceived direction of motion caused by the observer's own motion. The classic example of aberration involves two men out in the rain. One man is stationary and perceives the velocity of the rain as straight down from overhead at velocity **u**. The other man is walking along the ground at velocity **v**. In the moving man's frame, the velocity of the rain is **u-v.** (This is the value as computed using Galilean Relativity; the value according to Special Relativity is more complicated but the conclusions are the same). This relative velocity makes an angle φ with the vertical where

$$\varphi = \tan^{-1}\left(\frac{v}{u}\right) \tag{13}$$

The faster the man walks, the larger his perceived deflection of the rain from the vertical.

In technical sources, aberration is usually discussed in the context of either stellar or planetary aberration. Stellar aberration was first considered when looking at stars through optical telescopes – it is the perceived change in direction of light. Planetary aberration usually refers to two effects combined, light time delay and the perceived change in the direction of light. In both cases, the observer's velocity relative to the frame in which the light path was computed results in aberration.

Stellar Aberration

Typically, starlight is modeled as saturating the solar system with light. The light is considered to move in a straight line through the solar system. The actual transmission time at the star is unmodeled (being more uncertain than the direction to the star itself) so light time delay is not considered. However, aberration caused by an observer's motion in the solar system as the observer receives the light can be computed, and is referred to as stellar aberration. Let the direction to a star from an observer (accounting for proper motion of the star and parallax) be $\hat{\mathbf{e}}_{\mathbf{r}}$. Then the apparent direction of the star, accounting for stellar aberration, is:

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{e}}_{\mathbf{r}} + \boldsymbol{\beta}}{\left\|\hat{\mathbf{e}}_{\mathbf{r}} + \boldsymbol{\beta}\right\|}, \text{where } \boldsymbol{\beta} = \frac{\mathbf{v}}{c} \tag{14}$$

and **v** is the velocity of the observer with respect to the solar system barycenter frame. The formula above is the Galilean formula, equation (3.252-1)[1]; the Special Relativity formula is given by (3.252-3)[1] and is simply a use of the Lorentz transformation for velocities. The formula above is accurate to order β.

> NOTE: The stellar aberration formula above models the observer receiving a signal, not transmitting a signal.

Annual and Diurnal Aberration

While the concept of aberration is simple, its computation can be complicated, depending on which factors are considered for determining the observer's velocity **v** with respect to *F*. Astronomers have compartmentalized different aspects of the computation, coining terms for each aspect's contribution. The term 'annual aberration' is meant to identify the contribution of the observer's central body velocity in the solar system:

$$\mathbf{v} = \mathbf{v}_{\mathbf{cb}} + \mathbf{v}_{\mathbf{A/cb}} \tag{15}$$

where $\mathbf{v}_{\mathbf{cb}}$ is the velocity of the central body with respect to the solar system barycenter frame and $\mathbf{v}_{\mathbf{A/cb}}$ is the velocity of the observer A with respect to the central body. When $\mathbf{v}_{\mathbf{cb}}$ is used to compute aberration, rather than $\mathbf{v}$, then only the effects of annual aberration have been considered in the proper apparent relative position.

We term $\mathbf{v}_{\mathbf{A/cb}}$ the diurnal aberration (the contribution to $\mathbf{v}$ apart from the central body motion). In some technical sources, the term diurnal aberration is reserved for the contribution to $\mathbf{v}_{\mathbf{A/cb}}$ made by the rotation of the central body itself, and other terms are used to describe the other contributions to the overall value of $\mathbf{v}_{\mathbf{A/cb}}$.

Planetary Aberration

Usually, planetary aberration refers to two effects combined: light time delay and the stellar aberration (i.e., the change in the perceived direction of motion caused by an observer's motion). To order β, the results can be computed correctly using the simpler Galilean formulas.

We have previously discussed light time delay and determined a method for computing the light time delay Δt, the apparent relative position vector $\mathbf{r}$, and the signal path $\mathbf{s}$ by identifying an inertial frame F to perform the computations. We now consider the effect of aberration.

Consider another inertial frame F' coincident with the observer A at the event time t, whose constant velocity $\mathbf{v}$ is the value of the observer's velocity at time t. Because the observer's velocity is not (usually) constant in time, we'll associate a new inertial frame F' for each time t, calling the collection of inertial frames the co-moving inertial frames at A. The apparent position of B with respect to A as perceived by an observer at A at time t but moving with F' (computed by modeling the signal motion in F and then transforming this motion to F') is

$$\mathbf{r}_{\mathbf{p}} = \mathbf{r} - \sigma \Delta t\, \mathbf{v} = c\Delta t\left(\hat{\mathbf{e}}_{\mathbf{r}} - \sigma \boldsymbol{\beta}\right) = \mathbf{R}_{\mathbf{B}}(t + \sigma\Delta t) - \mathbf{R}_{\mathbf{A}}(t) - \sigma\Delta t \dot{\mathbf{R}}_{\mathbf{A}}(t) \tag{16}$$

where σ=1 when modeling a signal transmitted from A, and σ=-1 when modeling a signal received at A. Again, $\mathbf{r}$ is the apparent relative position of B with respect to A, so that the light path range r is $c\Delta t$ and β=$\mathbf{v}$/c. This formula generalizes equation (3.255-2)[1] to cases of transmission and reception. The vector $\mathbf{r}_{\mathbf{p}}$ is the proper apparent relative position of B with respect to A, where the term 'proper' indicates that this quantity is computed as perceived by A (really, by an observer at A moving in a co-moving inertial frame).

When the light time delay Δt is small, it is possible to construct alternate representations of planetary aberration that approximate the exact expression (16). Expanding $\mathbf{R}_{\mathbf{B}}$ in a Taylor series in time, we find:

$$\mathbf{R_B}(t+\sigma\Delta t) = \mathbf{R_B}(t) + \sigma\Delta t\dot{\mathbf{R}}_\mathbf{B}(t) + \cdots \quad (17)$$

Using (17) in (16), we find:

$$\mathbf{r_p} \doteq \mathbf{R_B}(t) - \mathbf{R_A}(t) + \sigma\Delta t\left\{\dot{\mathbf{R}}_\mathbf{B}(t) - \dot{\mathbf{R}}_\mathbf{A}(t)\right\} \quad (18)$$

which is equation (3.255-4)[1] generalized to both transmit and receive cases. Similarly, expanding $\mathbf{R_A}$ in a Taylor series in time, we find:

$$\mathbf{R_A}(t) = \mathbf{R_A}(t+\sigma\Delta t) - \sigma\Delta t\dot{\mathbf{R}}_\mathbf{A}(t+\sigma\Delta t) + \cdots \quad (19)$$

Using (19) in (16), we find:

$$\mathbf{r_p} \doteq \mathbf{R_B}(t+\sigma\Delta t) - \mathbf{R_A}(t+\sigma\Delta t) + \sigma\Delta t\left\{\dot{\mathbf{R}}_A(t+\sigma\Delta t) - \dot{\mathbf{R}}_\mathbf{A}(t)\right\} \quad (20)$$

that can be simplified to

$$\mathbf{r_p} \doteq \mathbf{R_B}(t+\sigma\Delta t) - \mathbf{R_A}(t+\sigma\Delta t) \quad (21)$$

when the last expression in (20) is small. [This will be small for small Δt and small acceleration of A.] This generalizes equation (3.255-3)[1] for transmit and receive cases.

The proper apparent direction is computed to be

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{e}}_\mathbf{r} - \sigma\boldsymbol{\beta}}{\left\|\hat{\mathbf{e}}_\mathbf{r} - \sigma\boldsymbol{\beta}\right\|} \quad (22)$$

that of course agrees with the value computed for stellar aberration in the case of reception. The proper apparent direction depends on Δt only through **r**. Also note that the proper apparent range r_p is not the same as the light path range r, nor is the proper apparent range the same in the case of transmission and reception. This is consistent with Special Relativity as distances in F and F' differ.

Optical Measurements

Optical observations of satellite position are made by measuring the apparent satellite location against known stars in the telescope field of view. Observations collected in this manner can be used in determining the orbit of the satellite. These observations are a function of the corrections which have been applied to the star positions. Typically these corrections include such effects as the proper motion and parallax of the stars. Star coordinate corrections may optionally include annual and diurnal aberration due to the motion of the observer. Effects not accounted for in the computation of the star coordinates must be accounted for separately in observation processing. For example, omission of diurnal aberration from the star positions requires a diurnal aberration correction in orbit determination. Regardless of the corrections made to the star catalog, orbit determination must also account for the motion of the satellite during the time it takes for light to travel from the satellite to the observer.

[1] Explanatory Supplement to the Astronomical Almanac, Ken Seidelmann, ed., 1992.